# RECOGNITION OF TOTAL OR PARTIAL SYMMETRY IN A COMPLETELY OR INCOMPLETELY SPECIFIED SWITCHING FUNCTION

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# RECOGNITION OF TOTAL OR PARTIAL SYMMETRY IN A COMPLETELY OR INCOMPLETELY SPECIFIED SWITCHING FUNCTION

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## 1. Introduction

At the stage of the structural synthesis of a switching circuit, knowledge of the symmetry properties of the switching function has proved to be very valuable. First, if we do not limit the structure of the circuit, the methods of designing the switching circuits that realize symmetric functions are very simple [1-3]; especially when a circuit is constructed on such elements as relays [1] or threshold elements [3]. If we are interested in obtaining a minimal-cost two-level switching circuit, constructed on conventional gate-type switching elements "or", "and", "not" the problem arises of determining the minimal normal sum-of-products form.

Resolving of this problem can be [4,5] essentially simplified, once we have the information that the function is symmetric. This fact is particularly significant, because as a result of Kazakov's work [6], the class of switching functions possibly processing a maximal number of prime implicants, potentially the most difficult to minimize, is included in the class of symmetric functions *. Symmetry information is also useful for showing equivalence between two multiple output switching functions. And most of the gates used today to realize switching circuits produce symmetric functions.

A number of works have been devoted to the problem of recognizing symmetry in switching functions [7-12]. The majority

* Assuming that the class of symmetric functions includes the functions symmetric with respect to the literals, i. e. unprimed or primed variables (or functions $f(x) = x$ and $f(x) = \bar{x}$ ).

of these works deal with recognizing total symmetry in a switching function. Mukhopadhyay [10] describes a method for determining the total or partial symmetry sets of a completely specified switching function which uses n or $\binom{n}{2}$ respectively so-called "decomposition charts". Schneider and Dietmeyer [12] describe a computer-oriented method for recognizing symmetry in a completely or incompletely specified (multiple output) switching function which involves performing certain operations on the rows of an array which represents the given function. This method deals with recognizing symmetry with respect to (unprimed) variables.

The present paper describes a method for recognizing total or partial symmetry with respect to the literals in a (single or multiple output) switching function which may be completely or incompletely specified. The method represents a new approach which is based on the use a certain two-dimensional topological model of a function, the so-called function image T(f). The use of this model allows the method to be easily applied in hand (for $n \leqslant 7$-8) as well as machine calculations.

## 2. Notation and Definitions of the Basic Terms

Let $\Omega_j = (\omega_1, \ldots, \omega_n), j \in \{0, 1, \ldots, 2^n - 1\}$, $\omega_i \in \{0, 1\}$, $i = 1, \ldots, n$, denote a sequence of values of the input variables $x_1, \ldots, x_n$ and assume further that $j = \sum_{i}^{n} \omega_i 2^{n-i}$.

Let $\Omega = \{\Omega_j\}$ be the set of all such sequences $\Omega_j$.

Let $Y_k = (\gamma_1, \ldots, \gamma_m)$, $k \in \{0, 1, \ldots, 3^n - 1\}$, $\gamma_i \in \{0, 1, *\}$, $i = 1, \ldots, n$, where $*$ represents an unspecified ("don't care") value, denote a sequence of values of the output variables $y_1, \ldots, y_n$ and assume further that $k = \sum_{i=1}^{m} \hat{\gamma} \cdot 3^{n-i}$, where $\hat{\gamma}_i = \begin{cases} \gamma_i, & \text{if } \gamma_i = 0, 1 \\ 2, & \text{if } \gamma_i = * \end{cases}$. Let $Y = \{Y_j\}$ be the set of all such sequences $Y_j$.

A multiple output switching function f is then defined as a mapping from $\Omega$ into $Y$ $(f : \Omega \to Y)$. For our considerations now we assume that $m = 1$, but later we will show how to extend the method for the case when $m > 1$.

Let $\Omega^0$, $\Omega^1$, $\Omega^*$ denote the sets of sequences $\Omega_j$ for which $f(\Omega_j) = 0, 1, *$ respectively. If $\Omega^* = \theta$, where $\theta$ is the empty set, then the function $f$ is completely specified and if $\Omega^* \neq 0$ then $f$ is incompletely specified. An incompletely specified function $f$ determines a set $\Phi(f) = \{f_i^*\}$ of completely specified functions $f_i^*$, $i = 0, 1, \ldots, 2^{c(\Omega^*)} - 1$, where $c(\Omega^*)$ is the cardinality of the set $\Omega^*$. These functions $f_i^*$ are determined by all possible partitions of the set $\Omega^*$ into subset $\Omega^{*0}$ and $\Omega^{*1}$, such that $f_i^*(\Omega^{*0}) = \{0\}$ and $f_i^*(\Omega^{*1}) = \{1\}$.

<u>Definition 1</u>: A function $f(x_1, \ldots, x_n)$ is said to be <u>symmetric with respect to the set of literals</u> $X = \{x_i^{6_i}\}_{i \in I}$ where $I \subseteq \{1, \ldots, n\}$, $6_i \in \{0, 1\}$ and $x_i^{6_i} = \begin{cases} x_i, & \text{if } 6_i = 1; \\ \bar{x}_i, & \text{if } 6_i = 0 \end{cases}$ if:

(1) if $\Omega^* = \theta$, then the function is invariant with respect to any permutation of the literals of the set $X$;

(2) if $\Omega^* \neq \theta$, then in the set $\Phi(f) = \{f_i^*\}$ there is at least one function $f_i^*$ which satisfies condition (1).

A set of literals $X$ with respect to which the function $f$ is symmetric we will call a <u>symmetry set of</u> $f$. If $X$ is a symmetry set of $f$ and $X$ is <u>maximal</u> under inclusion among the symmetry sets of $X$, then we will say simply that $X$ is a <u>maximal symmetry set</u>. Symmetry sets of $f$ of largest cardinality among the symmetry sets of $f$ will be denoted $X_m$. If $c(X_m) = n$, the function $f$ is said to be totally symmetric, if $1 < c(X_m) < n$, $f$ is said to be partially symmetric.

If $X = \{x_k^{6_k}, x_l^{6_l}\}$ is a symmetry set of the function $f$, then, according to definition 1, $x_k^{6_k}$ can be permuted with $x_l^{6_l}$ and $x_l^{6_l}$ with $x_k^{6_k}$ without changing the function $f$, if $f$ is completely specified or if $f$ is incompletely specified, without changing some functions $f_i^*$ in the set $\Phi(f)$. We will denote the set of all such functions $f_i^*$ by $\Phi_{k,l}$. The relation saying that the literal $x_k^{\sigma_k}$ is permutable with $x_l^{\sigma_l}$ without changing the function $f$ is written $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_f$. If $f$ is incompletely specified then $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_f$ will be written to mean that the set $\Phi_{k,l}$ defined above is not empty. If functions of the set $\Phi_{k,l}$ are determined we write $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_{\Phi_{k,l}}$. It is easy to prove that the relation $\sim$ is reflexive, symmetric and transitive, but in the case $\Omega^* \neq 0$ transitivity is understood as

$$\left(x_k^{\sigma_k} \sim x_l^{\sigma_l}\right)_{\Phi_{k,l}} \ \wedge \ \left(x_l^{\sigma_l} \sim x_m^{\sigma_m}\right)_{\Phi_{l,m}} \Rightarrow \ \left(x_k^{\sigma_k} \sim x_m^{\sigma_m}\right)_{\Phi_{k,l} \cap \Phi_{l,m}} \qquad (1)$$

When $X = \left\{x_1^{\sigma_1}, \ldots, x_m^{\sigma_m}\right\}$ is a symmetry set, a relation $(x_1^{\sigma_1} \sim \cdots \sim x_m^{\sigma_m})_f$ can be defined analogously to the above.

The elementary symmetric functions are:

$$S_0\left(x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\right) = x_1^{1-\sigma_1} x_2^{1-\sigma_2} \ldots x^{1-\sigma_n}$$

$$S_1\left(x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\right) = x_1^{\sigma_1} x^{1-\sigma_2} \ldots x^{1-\sigma_n} \vee x^{1-\sigma_1} x^{\sigma_2} x^{1-\sigma_3} \ldots x^{1-\sigma_n} \vee \ldots$$

$$\vdots$$

$$S_n\left(x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\right) = x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_n^{\sigma_n}$$

Each totally symmetric function is a sum of a set of elementary symmetric functions and can therefore be written in the form $S_A(X)$, where $X = \left\{x_i^{\sigma_i}\right\}_{i=1,\ldots,n}$ is a symmetry set of the function, and $A = \left\{a_i\right\}_{i=1,2,\ldots}$ is the set of indices of the elementary symmetric functions whose sum is the given function (the so-called a-numbers set). It is also well known that:

$$S_A(X) \equiv S_{\hat{A}}(\hat{X}) \qquad (2)$$

where $\hat{X} = \left\{ x_i^{\hat{\sigma}_i} \right\}$, $\hat{\sigma}_i = 1 - \sigma_i$ and $\hat{A} = \left\{ \hat{a}_i \right\}_{i=1,2,\ldots}$, $\hat{a}_i = n - a_i$ . The function $S_A(x_1, \ldots, x_n)$ is written $S_A^n$ .

Any switching function may be written in the form of the following matrix:

$$\hat{T}(f) = \left[ t_{v,h} \right]_{\substack{v=0,1,\ldots,2^{\left[\frac{n}{2}\right]}-1 \\ h=0,1,\ldots,2^{n-\left[\frac{n}{2}\right]}-1}}$$

where $t_{v,h} = f(\Omega_j)$, $j = v \cdot 2^{\left[\frac{n}{2}\right]} + h$ and $\left[\frac{n}{2}\right]$ is the entire part of $\frac{n}{2}$ . To the matrix $T(f)$ there correspond in a one-to-one fashion a certain topological model called the image $T(f)$ of the function. The image $T(f)$ is determined with the use of the $n$-variable logical diagram defined below.

Let us divide any rectangle into $2^{\left[\frac{n}{2}\right]}$ rows and $2^{n-\left[\frac{n}{2}\right]}$ columns according to the rules:

(1) In the first step we divide a rectangle into two rows with a horizontal line. In step $m$ each row obtained in step $m-1$ we divide into two rows. We execute $\left[\frac{n}{2}\right]$ steps.

(2) The steps $\left[\frac{n}{2}\right] + 1, \ldots, n$ are executed similarly as it was done in rule (1), but by division of the rectangle into columns with vertical lines.

The lines which divide the rectangle in step $i$ $(i=1,\ldots,n)$ we call the axes of the variable $x_i$ . The intersection of any row with any column we call a <u>cell of the logical diagram</u>, assuming that the cell does not include the points belonging to any axis or the perimeter of the rectangle. To set of cells lying above $x_1$ axis we assign the literal $\bar{x}_1$ and to those lying below the axis the literal $x_1$ . The axes $x_1, x_2, \ldots, x_i$, $i = 1, 2, \ldots, \left[\frac{n}{2}\right] - 1$, divide the rectangle into $2^i$ sets of cells. To the set consisting of the cells of all the top halves of these sets described above we assign the literal $\bar{x}_{i+1}$ and to set consisting of the cells of all the bottom halves we assign the literal $x_{i+1}$ .

To the set of cells lying on the left of the axis $x_{\left[\frac{n}{2}\right]+1}$ we assign the literal $\bar{x}_{\left[\frac{n}{2}\right]+1}$ and to the set of those lying on the right the literal $x_{\left[\frac{n}{2}\right]+1}$. The axes $x_{\left[\frac{n}{2}\right]+1}, \ldots$ $\ldots, x_{\left[\frac{n}{2}\right]+i}$, $i = 1,2,\ldots, n - \left[\frac{n}{2}\right] - 1$, divide the rectangle into $2^i$ set of cells. To the set consisting of the cells of all the left halves of those sets described above we assign the literal $\bar{x}_{\left[\frac{n}{2}\right]+i+1}$ and to the set consisting of the cells of all the right halves we assign the literal $x_{i+1}$.

<u>Definition 2</u>: The figure described above we call an <u>n-variable logical diagram</u> (see Fig. 1 *).

The n-variable logical diagram is similar to the "tables"** of Venn [13], "charts" of Veitch [14], "maps" of Karnaugh [15] and other diagrams. To the logical operations on the literals there correspond set-theoretic operations on the sets of cells assigned to these literals: to the product of literals there corresponds the intersection of sets of cells, to the sum of literals the union of sets of cells. The set of cells corresponding to a product $\alpha$ of literals we will denote by $L(\alpha)$. To a product of $n$ literals $\alpha = x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ corresponds in a one-to-one fashion one cell $e$ of the diagram. We have $\alpha = 1$, when the variables $x_i$, $i = 1, \ldots, n$, take values $\omega_i = \sigma_i$. Then the cell $e$ corresponds uniquely to the sequence $\Omega_j = (\omega_1, \ldots, \omega_n)$, where $\omega_i = \sigma_i$ and $j = \sum_{i=1}^{n} \omega_i 2^{n-i}$.

<u>Definition 3</u>. The <u>number of the cell</u> $e$ in an n-variable logical diagram is the number $\gamma(e) = j$, where $j$ is the index of the sequence $\Omega_j$ to which corresponds the cell $e$.

The cell numbers $\gamma(e)$ are distributed in the logical di-

* In this figure a given variable $x_i$ is written beside only one of the $x_i$ variable axes.

** Not to be confused with the "diagrams" of Venn.

agram in an ordered manner. For instance in Fig. 2 is shown the distribution of cell numbers in a 6-variable logical diagram.

Definition 4: The weight of the cell $e$ in an $n$-variable logical diagram is the number $\delta(e) = \sum_{i=1}^{n} \omega_i$, where $\omega_i$, $i = 1, \ldots, n$, are elements of the sequence $\Omega_{\gamma(e)}$.

The weights of the cells are located in the logical diagram in a certain characteristic order. Fig. 3 presents the example of the distribution of weights in a 6-variable logical diagram.

To each cell $e$ in an $n$-variable logical diagram let us assign the value $f(\Omega_{\gamma(e)})$ where $f(x_1, \ldots, x_n)$ is a given function.

Definition 5: The set of all cells of an $n$-variable logical diagram with values assigned as described above we call the image of the function $f$ and denote by $T(f)$.

If $e$ is a cell in $T(f)$ to which was assigned the value $p$, we will say simply that $e \in T(f)$ has value $p$. We define $F^p = \{e \in T(f) : e \text{ has value } p\}$, where $p = 0, 1, *$. Let

$$\alpha_1 = \bigwedge_{i \in I_1} x_i^{\sigma_i^1} \quad \text{and} \quad \alpha_2 = \bigwedge_{i \in I_2} x_i^{\sigma_i^2}$$

where $I_1 = I_2 \leqslant \{1, \ldots, n\}$.

Definition 6: A pair of cells $(e_j^1, e_j^2)$, where $e_j^1 \in L(\alpha_j)$ and $e_j^2 \in L(\alpha_2)$ are called a pair of corresponding cells in $L(\alpha_1)$ and $L(\alpha_2)$ if $\gamma(e_j^1) - \gamma(e_j^2) = \sum_{i \in I} (\sigma_i^1 - \sigma_i^2) 2^{n-i}$ where $I = I_1 = I_2$.

It can beseen that corresponding cells in $L(\alpha_1)$ and $L(\alpha_2)$, if $I_1 = I_2$, we can make coincide by a parallel shift (see Fig. 4).

## 3. Recognition of Symmetry in Completely Specified Functions

### 3.1. Recognition of Total Symmetry

A given function is totally symmetric if $c(X_m) = n$. Detection of that fact is based on the following theorem:

Theorem 1: A necessary and sufficient condition that a function $f(x_1, \ldots, x_n)$ be totally symmetric is the existence of a sequence $\sigma = (\sigma_2, \sigma_3, \ldots, \sigma_n)$, $\sigma_i \in \{0, 1\}$, such that

$$\forall (i \in \{2, \ldots, n\}), (x_1 \sim x_i^{\sigma_i})_f$$

Proof: Sufficiency: This results immediately from the fact that the relation $\sim$ is transitive. Necessity: According to formula (2), if the set $\{\bar{x}_1, x_2^{\sigma_2}, \ldots, x_n^{\sigma_n}\}$ is a symmetry set then the set $\{x_1, x_2^{1-\sigma_2}, \ldots, x_n^{1-\sigma_n}\}$ is also a symmetry set. Therefore if the function is totally symmetric, there exists the symmetry set including the literal $x_1$. Suppose then f is totally symmetric and $\{x_1, x_2^{\sigma_2}, \ldots, x_n^{\sigma_n}\}$ is a symmetry set of $f$. Then clearly $\sigma_2, \ldots, \sigma_n$ is a sequence with the desired property. Q.E.D.

This theorem is also valid if we permute in any way the indices $i \in \{1, 2, \ldots, n\}$ of the literals $x_i^{\sigma_i}$.

Theorem 1 shows that testing whether the function f is to-symmetric consists of testing for symmetry with respect to certain pairs of literals. The minimal number of pairs to test is $n - 1$ and maximal $2(n - 1)$.

Let us expand the function $f(x_1, \ldots, x_n)$ with respect to the variables $x_k$ and $x_l$, $k,l \in \{1, \ldots, n\}$, $k \neq n$:

$$f(x_1, \ldots, x_n) = \bar{x}_k\bar{x}_l f^0_{k,l} \vee \bar{x}_k x_l f^1_{k,l} \vee x_k\bar{x}_l f^2_{k,l} \vee x_k x_l f^3_{k,k} \quad (3)$$

where $f^i_{k,l} = f(x_1, \ldots, x_{k-1}, \omega^i_k, x_{k+1}, \ldots, x_{l-1}, \omega^i_l, x_{l+1}, \ldots, x_n)$, $i = 0, 1, 2, 3$, $i = 2\omega^i_k + \omega^i_l$.

Theorem 2: A. $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_f$ where $\sigma_k \neq \sigma_l$; $\sigma_k, \sigma_l \in \{0,1\}$ if and only if $f^1_{k,l} \equiv f^2_{k,l}$ (a)

B. $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_f$ where $\sigma_k \neq \sigma_l$; $\sigma_k, \sigma_l \in \{0,1\}$ if and only if $f^0_{k,l} \equiv f^0_{k,l}$ (b)

Proof:

A. If (a) is satisfied by transformation of the formula (3) we obtain:

$$f(x_1, \ldots, x_n) = \bar{x}_k\bar{x}_l f^0_{k,l} \vee (\bar{x}_k x_l \vee x_k\bar{x}_l) f^1_{k,l} \vee x_k x_l f^3_{k,l}$$

This equation shows that permutations $\begin{pmatrix} x_k, x_l \\ x_l, x_k \end{pmatrix}$ and $\begin{pmatrix} \bar{x}_k, \bar{x}_l \\ \bar{x}_l, \bar{x}_k \end{pmatrix}$ do not change the function $f$, that is $(x_k^{\sigma_k} \sim x_l^{\sigma_l})_f$ in case $\sigma_k = \sigma_l$. If (a) is not satisfied, then the above permuta - tions change $f$, proving the necessity of (a).

B. This can be proved analogously.

Q.E.D.

Conditions (a) and (b) can be easily checked using the image $T(f)$. The images $T(f^i_{k,l})$, $i = 0,1,2,3$, are determined by the subsets of the image $T(f)$ consisting of the cells of $L^i = L(x_k^{\sigma_k^i} x_l^{\sigma_l^i})$, where $i = 2\sigma_k^i + \sigma_l^i$. Such subsets of $T(f)$ we will denote by $T^i_{k,l}(f)$. Condition (a), (b) is then equivalent to the relation $T^1_{k,l}(f) \doteq T^2_{k,l}(f)$, $(T^0_{k,l}(f) \doteq T^3_{k,l}(f))$, where by $\doteq$ we mean that corresponding * cells in $T^1_{k,l}(f)$ and $T^2_{k,l}(f)$ ($T^0_{k,l}(f)$ and $T^3_{k,l}$) have the same value. When the condition (a) is satisfied we say that $f$ has symmetry of the first kind and when condition (b) is satisfied that $f$ has symmetry of the second kind with respect to the pair of variables $\{x_k, x_l\}$.

Fig. 5 presents algorithm S for testing the total symmetry of the function $f$ based on theorems 1 and 2. The sign := is used as in Algol (it denotes that the variable on the left of the sign takes the new value resulting from the operation written on the right). As a result of algorithm S, we obtain the answer whether the function $f$ is totally symmetric and in the case of a positive answer, a symmetry set $X_m$ is determined. In general the function $f$ may have a number of alternative symmetry sets $X^i_m$, $i = 1, 2, \ldots$, whose difference from one another is that contain different litterals of the

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* By corresponding cells in $T^{p_1}_{k,l}(f)$ and $T^{p_2}_{k,l}$ ($p_1 = 1, 0$, $p_2 = 2,3$) we mean corresponding cells in $L^{p_1}$ and $L^{p_2}$.

same variables. To determine all sets $X_m^i$, we modify algorithm S , so that both condition (a) and(b) are tested for each pair of variables $(x_1, x_i)$ , $i = 2, 3, \ldots, n$.

In the case of hand realization of the algorithm it is more convenient to adopt a different order of testing for symmetry with respect to pairs of variables, namely to test for symmetry with respect to pairs: $(x_1, x_{\left[\frac{n}{2}\right]+1})$, $(x_1, x_{\left[\frac{n}{2}\right]+2})$, $\ldots$, $(x_1, x_n)$, $(x_2, x_{\left[\frac{n}{2}\right]+1})$, $(x_3, x_{\left[\frac{n}{2}\right]+1})$, $\ldots$, $(x_{\left[\frac{n}{2}\right]}, x_{\left[\frac{n}{2}\right]+1})$. In this case the sets of cells whose values we compare to test for symmetry are characteristically located in the diagram and are easy to define (see Fig. 6).

## 3.2. Determination of the a-numbers Set

If we have determined that a function f is totally symmetric then we can represent it in the form $S_A(X)$, where A is the set of a-numbers and X is the symmetry set. Having determined that f is totally symmetric, we will also have found a symmetry set $X = X_m$. Thus we need only to determine the set A in order to find the form $S_A(X) \equiv f$.Determination of this set is helped by the following theorem:

Theorem 3: Let f be a function such that $f \equiv S_A(X)$, $X = \{x_1, x_2, \ldots, x_n\}$, $A \subseteq \{0, 1, \ldots, n\}$. The set $F^1$ is the set of all cells $e \in T(f)$ having weights $\delta(e) \in A$ .

Proof: Recall that $F^1$ is the set of all cells in T(f) with value 1. Let $A = \{a_i\}_{i=1,2,\ldots}$. Because $S_A(X) \equiv f$ then $f = \bigvee_{a_i \in A} S_{a_i}(X)$. A function $S_{a_i}(X)$, $a_i \in A$ has $\binom{n}{a_i}$ components, each including $a_i$ literals $x_i$ and $n-a_i$ literals $\bar{x}_i$ . To each component there corresponds the sequence $\Omega_j$ including $a_i$ ones and $n - a_i$ zeros, so the weights of the cells corresponding to those sequences are $\delta(e) = a_k$. As the function f is a sum of $S_{a_i}$ , $i = 1, 2, \ldots$, then $F^1$ consists of all cells, which have weights $\delta(e) \in A$. Q.E.D.

Theorem 3 is illustrated by Fig. 6, which shows the image of the function $S^6_{2,3}$ (the cells of $F^1$ are dark). Due to this theorem the determination of the a-numbers in case the symme -

try set is $\{x_i\}$, $i = 1, \ldots, n$, may be accomplished by the next operations:

1. Determination of a certain minimal set of cells $E = \{e_i\}$ such that the weights $\delta(e_i)$ exhaust the set of all possible values of the a-numbers i.e. the set $\{0, 1, \ldots, n\}$.

2. The test showing which cells $e_i$ belong to $F^1$. If $e_i \in F^1$ then $a_i = \delta(e_i) \in A$.

It can be easily proved that $E = E_h \cup E_v$, where $E_h$ consists of the cells $e$, which have numbers $\gamma(e) = 2^k - 1$, $k = 0, 1, \ldots, n - \left[\frac{n}{2}\right]$ and $E_v$ consists of the cells $e$, which have numbers $\gamma(e) = 2^{n-\left[\frac{n}{2}\right]+k} - 1$, $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$. The set $E_h$ is included in the set $H = L(\bar{x}_1, \ldots, \bar{x}_{\left[\frac{n}{2}\right]})$ consisting of the cellslying in the top row of the diagram. The set $E_v$ is included in the set $V = L(x_{\left[\frac{n}{2}\right]+1}, \ldots, x_n)$ consisting of the cells lying in the right hand column of the diagram. Figure 7 shows the set $E = E_h \cup E_v$ in the 6-variable logical diagram.

Let us now assume that the symmetry set $X_s = \{x_i^{\sigma_i}\}$, $i = 1, 2, \ldots, n$, includes at least one literal $\bar{x}_i$. To reduce the present case to the previous one we might construct an image $T'(f)$ which consists of the cells of a logical diagram, whose variable axes are $x_i'$, $i = 1, \ldots, n$, where $x_i' = x_i^{\sigma_i}$.

The image $T'(f)$ can be constructed from $T(f)$ by successively replacing the sets of cells in $T(f)$ which were assigned the literal $x_i$ by those which were assigned the literal $\bar{x}_i$ and vice versa, for each $i$ such that $\sigma_i = 0$.

For determination of the set A we need only test the values of the cells of $E$, which are included in the set of cells of top row $H$ and right hand column $V$ of $T'(f)$. Therefore it is enough to perform replacements described above inside the set $L(x_1^{1-\sigma_1}, \ldots, x_{\left[\frac{n}{2}\right]}^{1-\sigma_{\left[\frac{n}{2}\right]}})$ for literals $\bar{x}_i \in X$, $\left[\frac{n}{2}\right]+1 \leqslant$

$\leqslant i \leqslant n$ and inside the set $L(x_{\left[\frac{n}{2}\right]+1}, \ldots, x_n^{\sigma_n})$ for literals $\bar{x}_i \in X$, $1 \leqslant i \leqslant \left[\frac{n}{2}\right]$. The sets so obtained will be the top row and right hand column respectively of the image $T'(f)$ which we seek in order to determine $A$.

## 3.3. Recognition of Partial Symmetry

A given function $f(x_1, \ldots, x_n)$ is partially symmetric if $1 < c(X_m) < n$. Determination of all the maximal symmetry sets $X^i$, $i = 1, 2, \ldots,$ of function $f$ may be easily accomplished when all symmetry pairs are known. The following implication is helpful here:

$$(\tilde{X}_a \sim x_p^{\sigma_p})_f \wedge (\tilde{X}_b \sim x_p^{\sigma_p})_f \Rightarrow (\tilde{X}_a \sim \tilde{X}_b \sim x_p^{\sigma_p})_f \qquad (4)$$

where $\tilde{X}_a$ represents a literal $x_a$ or a sequence of literals connected by sign $\sim$ : $x_{a_1}^{\sigma_{a_1}} \sim x_{a_2}^{\sigma_{a_2}} \sim \ldots$ and $\tilde{X}_b$ - analogously.

This implication results immediately from the transitivity of the relation $\sim$ . Fig. 9 presents algorithm $S^p$ for the determination of all the symmetry pairs of the function f. Because the relation $\sim$ is transitive, certain operations of this algorithm may be omitted: if $(x_a^{\sigma_a} \sim x_b^{\sigma_b})_f$ and $(x_a^{\sigma_a} \sim x_c^{\sigma_c})_f$ then also $(x_b^{\sigma_b} \sim x_c^{\sigma_c})_f$ and testing for symmetry with respect to the pair $\{x_b^{\sigma_b}, x_c^{\sigma_c}\}$ is superfluous. When f is totally symmetriç, algorithm $S^p$ is equivalent to algorithm S, in which for each pair of variables both conditions (a) and (b) are tested.

## 3.4. Determination of an Algebraic Form of a Partially Symmetric Function

Let $X = \{x_i^{\sigma_i}\}_{i \in I}$ where $I \subset \{1, \ldots, n\}$ be a symmetry set of the function $f(x_1, \ldots, x_n)$. Let us assume,without loss of generality that $I = \{1, 2, \ldots, m\}$, $1 < m < n$. Generalizing Shannon's [16] formula to the case of symmetry with respect to literals, the function $f(x_1, \ldots, x_n)$ may be written:

$$f(x_1, \ldots, x_n) = \bigvee_{j=1}^{m} S_j(X) f_j(x_{m+1}, x_{m+2}, \ldots, x_n) \qquad (5)$$

where $f_j(x_{m+1}, x_{m+2}, \ldots, x_n) = f(\omega_1, \ldots, \omega_m, x_{m+1}, x_{m+2}, \ldots$ $\ldots, x_n)$, $\omega_i = \begin{cases} 1, & \text{if } x_i \in X \\ 0, & \text{if } \bar{x}_i \in X \end{cases}$, for $i = 1, 2, \ldots, j$, $\omega_i =$ $= \begin{cases} 0, & \text{if } x_i \in X \\ 1, & \text{if } \bar{x}_i \in X \end{cases}$, for $i = j+1, j+2, \ldots, m$,

If we have found the maximal symmetry sets $X^i$ of the function $f(x_1, \ldots, x_n)$ for $X^i$ we can determine the form (5). Functions $f_j$ may be determined from any algebraic form of the function $f$.

## 4. Recognition of Symmetry in Incompletely Specifies Functions

Let $f(x_1, \ldots, x_n)$ be an incompletely specified function. The function $f$ is symmetric with respect to the pair $(x_k^{\sigma_k}, x_l^{\sigma_l})$. if there exists a non-empty set $\Phi_{k,l} \subseteq \Phi(f)$, such that for each $f_i^* \in \Phi_{k,l}$

(a) $T_{k,l}^1(f_i^*) \doteqdot T_{k,l}^2(f_i^*)$ , if $\sigma_k = \sigma_l$

or

(b) $T_{k,l}^0(f_i^*) \doteqdot T_{k,l}^3(f_i^*)$ , if $\sigma_k \neq \sigma_l$

Let us assume that $\sigma_k = \sigma_l$. Let $\{e_j^k, e_j^l\}$ , $j = 1, 2, \ldots$ $\ldots, 2^{n-2}$, $e_j^k \in T_{k,l}^1(f_i^*)$, $e_j^l \in T_{k,l}^2(f_i^*)$ be the sets corresponding cells in $T_{k,l}^1(f_i^*)$ and $T_{k,l}^2(f_i^*)$ . The set $\Phi_{k,l}$ is determined (with a certain restriction - see row 3 in the table below) by an unspecified function $f_{k,l}$ , whose image $T(f_{k,l})$ results from the image $T(f)$ by realization on each pair $\{e_j^k, e_j^l\}$ of the following operations which depend on the values of the cells of this pair:

| | $e_j^k$ | $e_j^l$ | |
|---|---|---|---|
| 1 | 0 | 0 | values of $e_j^k$ and $e_j^l$ are not changed |
| 2 | 1 | 1 | |
| 3 | $*$ | $*$ | values are not changed, but for each $f_i^* \in \Phi_{k,l}$, $e_j^k$ and $e_j^l$ must have the same value 0 or 1 |
| 4 | 0 | $*$ | values $*$ are changed to 0 or 1 to obtain equality of values of $e_j^k$ and $e_j^l$ |
| 5 | $*$ | 0 | |
| 6 | 1 | $*$ | |
| 7 | $*$ | 1 | |
| 8 | 0 | 1 | contradiction, $\Phi_{k,l} = \Theta$ |
| 9 | 1 | 0 | |

From this table we see, that a necessary condition for symmetry with respect to the pair $\{x_k^{\sigma k}, x_l^{\sigma l}\}$ is that the cells of no pair $\{e_j^k, e_j^l\}$ have values 0 and 1 or 1 and 0 respectively. In case $\sigma_k \neq \sigma_l$ the symmetry condition is analogous , but with respect to the pairs of corresponding cells in $T_{k,l}^0(f)$ and $T_{k,l}^3(f)$.

From the above considerations it follows that for a given image $T(f)$ and values of $\sigma_k$ and $\sigma_l$ the set $\Phi_{k,l}$ can be uniquely determined from the ordered triple $(C_{k,l}^0, C_{k,l}^1, R_{k,l})$ where $C_{k,l}^0$ is the set of cells whose values change from $*$ (in $T(f)$) to 0 (in $T(f_{k,l})$), $C_{k,l}^1$ is the set of cells whose values change from $*$ to 1 and $R_{k,l}$ is the family of sets $\{e_j^k, e_j^l\}$ such that cells $e_j^k$ and $e_j^l$ of each set have in $T(f)$ value $*$ . The cells of every set $\{e_k^j, e_l^j\} \in R_{k,l}$ in the image of each $f_i^* \in \Phi_{k,l}$ must have the same value 0 or 1 .

Let us assume that $(x_k^{\sigma k} \sim x_l^{\sigma l})_f$, $(x_l^{\sigma l} \sim x_m^{\sigma m})_f$ and that the triples $(C_{k,l}^0, C_{k,l}^1, R_{k,l})$ and $(C_{l,m}^0, C_{l,m}^1, R_{l,m})$ have been determined.

<u>Theorem 4</u>: $(x_k^{\sigma k} \sim x_l^{\sigma l})_f \wedge (x_l^{\sigma l} \sim x_m^{\sigma m})_f \Rightarrow (x_k^{\sigma k} \sim x_l^{\sigma l} \sim x_m^{\sigma m})_f$ if and only if the following are true:

(a) $C_{k,l}^0 \cap C_{l,m}^1 = \Theta$

(b) $C_{k,l}^1 \cap C_{l,m}^0 = \Theta$

(c) $\nexists\, (c^0, c^1) \in (C^0_{k,l} \times C^1_{k,l} \cup C^0_{l,m} \times C^1_{l,m}) \ni$
$$\nexists (C^*_i \in R_{k,l} \cup R_{l,m}), \{c^0, c^1\} \subseteq C^*_i$$

<u>Proof:</u>
Necessity: Let us assume that $(x_k^{\sigma k} \sim x_l^{\sigma l} \sim x_m^{\sigma m})_f$. Then there exists a non-empty $\Phi_{k,l,m} \subseteq \Phi(f)$ such that for each $f^*_i \in \Phi_{k,l,m}$ the set $\{x_k^{\sigma k}, x_l^{\sigma l}, x_m^{\sigma m}\}$ is a symmetry set. According to (1) $\Phi_{k,l,m} = \Phi_{k,l} \cap \Phi_{l,m}$. If (a) or (b) is not true, there exists $e$, which in the image of every $f^*_i \in \Phi_{k,l}$ has the oposite value than in the image of every $f^*_i \in \Phi_{l,m}$. From this it follows that $\Phi_{k,l} \cap \Phi_{l,m} = \theta$ and $\Phi_{k,l,m} = 0$, contrary to our assumption. If (c) is not true, there exists a pair of cells, which for each $f^*_i \in \Phi_{k,l}$ have opposite and for each $f^*_i \in \Phi_{l,m}$ have equal values or vice versa.

From this also follows that $\Phi_{k,l,m} = 0$.
Sufficiency: The sufficiency of conditions (a), (b), (c) we will prove by the construction of $\Phi_{k,l,m}$.
Let $C^0_{k,l,m} = C^0_{k,l} \cup C^0_{l,m} \cup C^0_r$ and $C^1_{k,l,m} = C^1_{k,l} \cup C^1_{l,m} \cup C^1_r$ where $C^0_r$ is the set of all cells $e^0_1$ such that
$$\forall e^0_1 \exists (e^0_2 \in C^0_{k,l} \cup C^0_{l,m}), \{e^0_1, e^0_2\} \in R_{k,l} \cup R_{l,m}$$
$C^1_r$ is the set of all cells $e^1_1$ such that
$$\forall e^1_1 \exists (e^1_2 \in C^1_{k,l} \cup C^1_{l,m}), \{e^1_1, e^1_2\} \in R_{k,l} \cup R_{l,m}$$

Let $R^0_{k,l}, R^1_{k,l} \subseteq R_{k,l}$ denote the sets containing those members of $R_{k,l}$ which contain the cells $e^0_1$ and $e^0_2$ respectively. Let $R^0_{l,m}$ and $R^1_{l,m}$ denote analogous subsets of $R_{l,m}$.
Let $R_1 = R_{k,l} \setminus (R^0_{k,l} \cup R^1_{k,l})$ and $R_2 = R_{l,m} \setminus (R^0_{l,m} \cup R^1_{l,m})$.
Let $R_{k,l,m} = R_1 \sqcup R_2$, where $\sqcup$ denotes the following two step operation:

1. The set $R_1 \cup R_2$ is determined.
2. Any pair of members of $R_1 \cup R_2$ whose intersection is non-empty we combine to form a new single member which is the

sum of these members. We then repeat this operation on the newly obtained set in successive such steps until we obtain a set each two of whose members have void intersection. (In the case we now consider this process ends after one step).

If (a) and (b) than $(C^0_{k,l} \cup C^0_{l,m}) \cap (C^1_{k,l} \cup C^1_{l,m}) = \theta$ (i). If (c) than $C^0_r \cap (C^1_{k,l} \cup C^1_{l,m}) = 0$ (ii) and $C^1_r \cap (C^0_{k,l} \cup \cup C^0_{l,m}) = \theta$ (iii).

Because the members of $R_{k,l} \cup R_{l,m}$ including any cells of $C^0_{k,l} \cup C^1_{k,l} \cup C^0_{l,m} \cup C^1_{l,m}$ are disjoint then $C^0_r \cap C^1_r = \theta$ (iv). From (i) - (iv) it follows that

$$C^0_{k,l,m} \cap C^1_{k,l,m} = 0$$

Let $T(f_{k,l,m})$ denote the image of $f_{k,l,m}$ we get after changing in $T(f)$ the values of cells $C^0_{k,l,m}$ from $*$ to 0 and values of cells $C^1_{k,l,m}$ from $*$ to 1. Let $\{f^*_i\}_{i \in I_{k,l,m}}$, $I_{k,l,m} \subseteq \{0, 1, 2, \ldots, 2^{c(\Omega^*)} - 1\}$ be a set of completely specified functions whose images $T(f^*_i)$ we get by changing the values $*$ of cells of $T(f_{k,l,m})$ in every possible way to 0 or 1, but assuming that to cells of every set of the family $R_{k,l,m}$ we assign the same value 0 or 1. For each function $f^*_i$, $i \in I_{k,l,m}$ we have the relations $(x^{\sigma_k}_k \sim x^{\sigma_l}_l)_{f^*_i}$ and $(x^{\sigma_l}_l \sim \sim x^{\sigma_m}_m)_{f^*_i}$. From this it follows that $(x^{\sigma_k}_k \sim x^{\sigma_l}_l \sim x^{\sigma_m}_m)_{f^*_i}$. Then we have $\Phi_{k,l,m} = \{f^*_i\}_{i \in I_{k,l,m}}$. If there are no cells in $T(f_{k,l,m})$ having value $*$ than $\Phi_{k,l,m}$ consists of only one function $f_{k,l,m}$. Q.E.D.

From the above proof of sufficiency of theorem 4 it results that the set $\Phi_{k,l,m}$ is uniquely determined by the triple $(C^0_{k,l,m}, C^1_{k,l,m}, R_{k,l,m})$. Theorem 4 gives conditions for obtaining three element symmetry sets from two two element symmetry sets whose intersection is non-void. We now generalize this theorem as shown below.

Let $A = \{a_1, \ldots, a_p\}$, $B = \{b_1, \ldots, b_r\}$, $A, B \subseteq \{1, \ldots, n\}, p, r \geqslant$

$\geqslant 2$, and $A \cap B \supseteq \{k\}$. Let $\widetilde{X}_A$ and $\widetilde{X}_B$ be sequences of literals connected by : $x_{a_1}^{\sigma a_1} \sim \ldots \sim x_{a_p}^{\sigma a_p}$ and $x_{b_1}^{\sigma b_1} \sim \ldots \sim x_{b_r}^{\sigma b_r}$

respectively. Let us assume without loss of generality that $a_1 = b_1 = k$. Now assume that $(\widetilde{X}_A)_f$ and $(\widetilde{X}_B)_f$ and that we have determined the corresponding triples $(C_A^0, C_A^1, R_A)$ and $(C_B^0, C_B^1, R_B)$.

<u>Theorem 5</u>: $(\widetilde{X}_A)_f \wedge (\widetilde{X}_B)_f \Rightarrow (\widetilde{X}_{A \cup B})_f$ if and only if

(a) $C_A^0 \cap C_B^1 = \theta$

(b) $C_A^1 \cap C_B^0 = \theta$

(c) $\nexists ((c^0, c^1) \in C_A^0 \times C_A^1 \cup C_B^0 \times C_B^1) \exists (C_i^* \in R_A \cup R_B)$ , $\{c^0, c^1\} \subseteq C_i^*$ .

Theorem 5 can be proved analogously to theorem 4. As in the proof of sufficiency in theorem 4 we can determine a triple $(C_{A \cup B}^0, C_{A \cup B}^1, R_{A \cup B})$. This triple describes the set $\phi_{A \cup B} \subseteq \subseteq \phi(f)$ of completely specified functions for which $(\widetilde{X}_{A \cup B})_{\phi_{A \cup B}}$.

As we have seen above it is easy to determine the symmetry pairs $(x_i^{\sigma i}, x_j^{\sigma j})$ and their corresponding triples $(C_{i,j}^0, C_{i,j}^1, R_{i,j})$ using the image T(f). Once knowing all such pairs and triples for f we can by virtue of Theorem 5 compute all maximal symmetry sets of f and their corresponding triples. It is also possible to compute the maximal symmetry sets of f using only the image T(f) (by making the proper assigments of the values * in T(f) for each symmetry pair we test in using algorithm $S^P$ ). But in this case we must use one „copy" of the image T(f) for each maximal symmetry set so obtained.

Let us assume now that f is an m-output incompletely specified function. It is then equivalent to the set of one-output functions $\{y_j\} = \{f_j(x_1, \ldots, x_n)\}$, $j = 1, \ldots, m$. If we determine the symmetry sets $\{X_j^i\}$ then, as it is easy to prove [12], all symmetry sets of f are the intersections of symmetry sets picked in all possible ways from each $\{X_j^i\}$, $j = 1, \ldots, m$.

## 5. Conclusions

The method for recognition of symmetry described in this paper may be easily performed by hand calculations for completely specified functions up to 7-8 variables. In the case of incompletely specified functions because in addition we must assign values for * , the practically of this method depends also on the cardinality of $\Omega^{*}$ .

In machine calculations this method can be carried out by the use of matrices and submatrices corresponding to the images T(f) and $T^{i}_{k,l}(f)$, i = 0, 1, 2, 3 respectively. Operations involving the comparison of matrices which are used in this method are suitable for digital computers, especially of the parallell type.

At the end it is worth noting that the notion of the image T(f) described in this paper is useful also for analysis and synthesis of any type of switching circuits, in particular (as was partly shown in work [17]) for the synthesis of minimal forms of switching functions.

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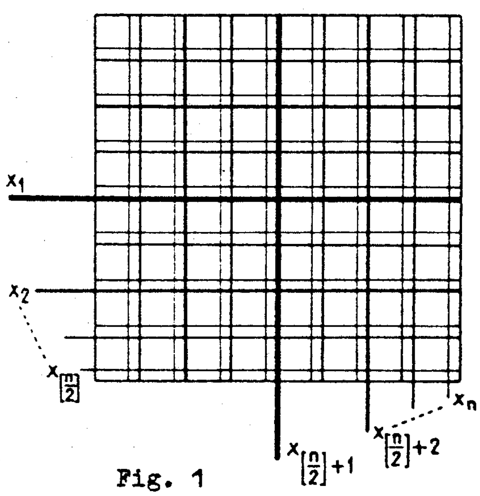


Fig. 1

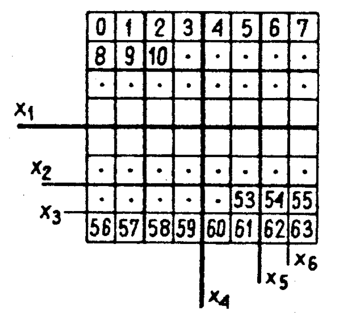


Fig. 2

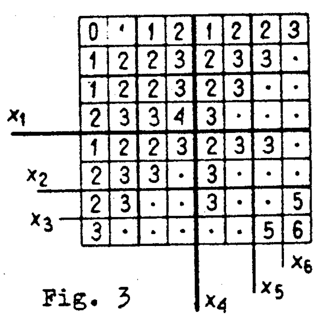


Fig. 3

$\alpha_1 = \bar{x}_1 x_5$

$\alpha_2 = x_1 \bar{x}_5$

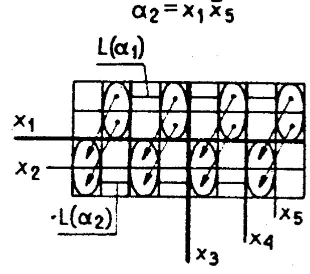


Fig. 4

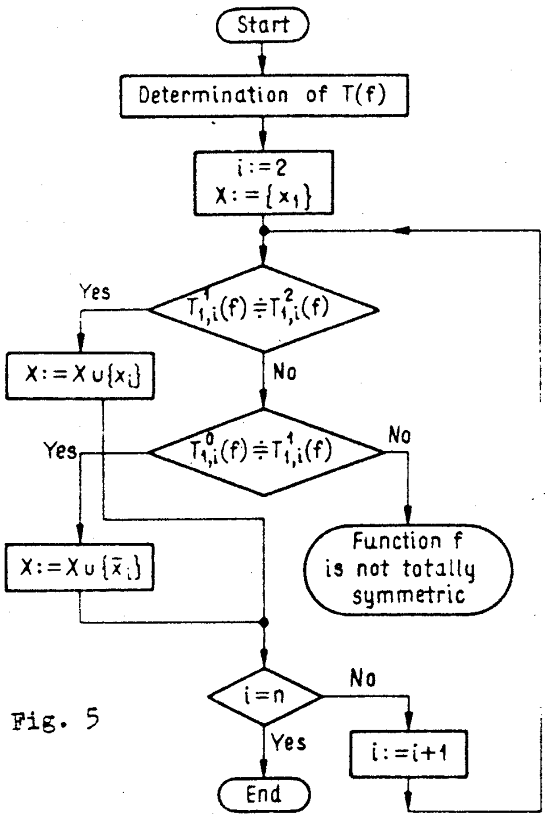
Start
Determination of T(f)
i:=2
X:={x1}
$T^1_{1,i}(f) \doteqdot T^2_{1,i}(f)$
Yes
No
X:=X ∪ {xi}
$T^0_{1,i}(f) \doteqdot T^1_{1,i}(f)$
Yes
No
X:=X ∪ {x̄i}
Function f is not totally symmetric
i=n
No
Yes
i:=i+1
End

Fig. 5

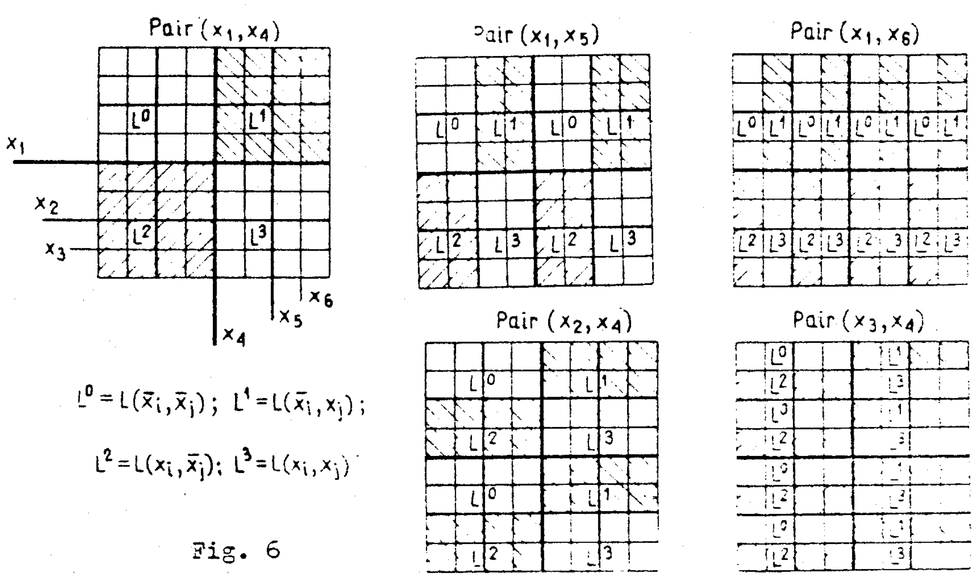
Pair (x1, x4)
Pair (x1, x5)
Pair (x1, x6)
Pair (x2, x4)
Pair (x3, x4)
x1
x2
x3
x4
x5
x6
L0
L1
L2
L3
$L^0 = L(\bar{x}_i, \bar{x}_j)$; $L^1 = L(\bar{x}_i, x_j)$;
$L^2 = L(x_i, \bar{x}_j)$; $L^3 = L(x_i, x_j)$

Fig. 6

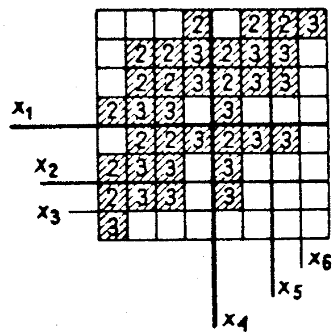


Fig. 7

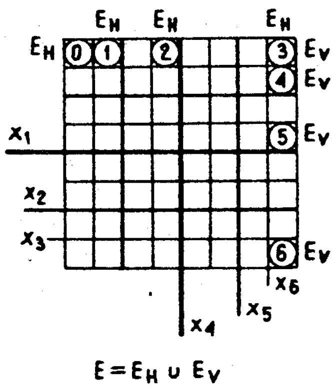


Fig. 8

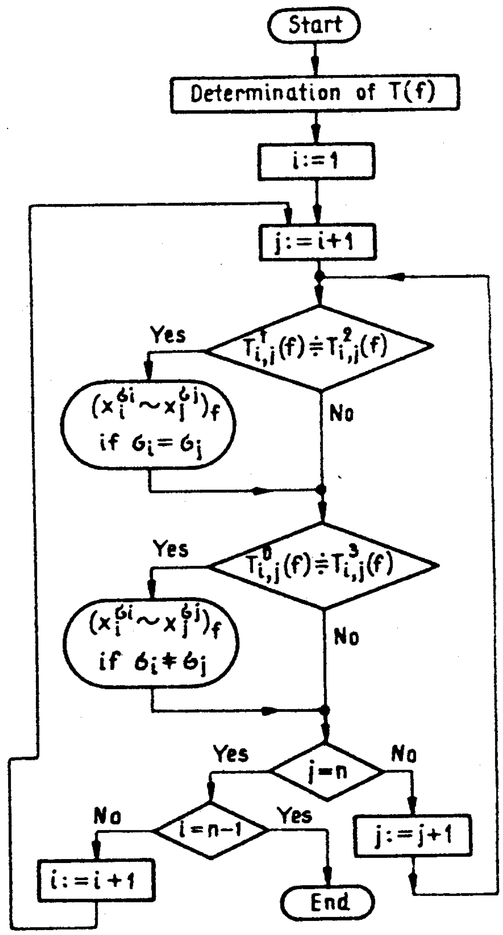


Fig. 9