INTERVAL GENERALIZATION
OF SWITCHING THEORY

by

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ABSTRACT

The paper considers:
(1) a Boolean algebra $< 2^E, \cup, \wedge, -, E, \emptyset >$ of event sets E_i from a discrete finite vector space E, and
(2) mappings f from the set E into $\{[0,1], *\}$, where * represents some unspecified value. A special case of the above is the Boolean algebra and Boolean functions considered in switching theory, where E is a space of binary vectors and f maps E into $\{0,1,*\}$, i.e. into the endpoints of the interval [0,1] and *.

A meet semi-lattice of multidimensional intervals (interval complexes) in E is introduced and then the concepts of exact, free, unordered and ordered interval covers of f are defined.

The simplest case of a cover - an unordered exact cover of a set $F^{1\lambda}$ against $F^{0\lambda}$ - is defined as a set of interval complexes whose set-theoretic union covers a given subset $F^{1\lambda} \subseteq E$ (defined as $\{ e \mid f(e) \geq \lambda \}$) and does not cover any element of another given subset $F^{0\lambda} \subseteq E \setminus F^1$.

The concept of ordered covers was developed to accomodate a preferential order in covering the set of 'mixed' events, defined as $\{ e \mid 0 < f(e) < 1 \}$ (a case not considered in classical switching theory).

The synthesis algorithm of covers is based on the 'method of disjoint stars', which has proved to be very useful for synthesis of complex switching systems. Quasi-minimal covers, produced by this method, are either minimal or approximately minimal. However, when we cannot state that the obtained solution is minimal, an estimate of its maximal possible distance to the minimum is provided.

Applications of the interval coverings concepts to pattern recognition and picture filtering are delineated.

1. INTRODUCTION

Some concepts and methods, initially developed for switching theory purposes, seem to have more universal application, if properly generalized. The generalization described in this paper stems from three observations:

(1) The coverings arising in the minimization of switching circuits can be viewed as a limiting case of interval coverings, introduced in the paper.

(2) The so-called method of disjoint stars, which was originally
     developed to provide a minimal or quasi-minimal solution of
     the covering problem in switching theory [1,3,4], and then
     extended to provide the quasi-minimal solution of the general
     covering problem [2], can be applied in particular to the
     synthesis of the above-mentioned interval coverings.

(3) The concept of a covering can be extended in yet another
     direction to accommodate a preferential ordering of the elements
     to be covered.

A need for the generalization described in the paper first
appeared when we were considering some problems of pattern recognition
and signal detection theory.

2. NOTATION AND DEFINITION OF AN INTERVAL COMPLEX

Elements of a discrete finite vector space E will be referred
to as events $e^j = (x_1, \ldots, x_n)$. Components x_i take their values from
the sets $\{0,1,2,\ldots,h_i-1\}$, $i = 1,2,\ldots,n$. We will assume that the
index j is given by

$$j = x_n + \sum_{i=1}^{n-1} \left(x_{n-i} \prod_{k=0}^{i-1} h_{n-k} \right) ; \qquad (1)$$

thus, as can be verified, j uniquely determines a vector e^j. The
value j will be called the <u>number</u> of the <u>event</u> e^j. When an event $e \epsilon E$
is given, its number will be denoted by $\gamma(e)$. For example, if
$e = (2,2,5,1)$, assuming that $h_1 = 5$, $h_2 = 3$, $h_3 = 3$ and $h_4 = 2$,
$\gamma(e) = 1 + 5(2) + 2(2.3) + 2(2.3.3) = 59$.

$E_1 \cup E_2$ and $E_1 \cap E_2$ (or $E_1 E_2$) will denote the set-theoretic
union and intersection of E_1 and E_2 respectively, where E_1 and E_2 are
sets of events. $\overline{E}_i$ is the complement of an event set E_i, defined as
$E \setminus E_i$, where $\setminus$ is the set-theoretic subtraction. The cardinality of
a set L will be denoted by $c(L)$.

<u>Definition</u> 1. By $^{a_i}X_i^{b_i}$, called a <u>literal</u>, we will denote the set
of all events $(x_1, \ldots, x_n)$ from the space E such that $a_i \leq x_i \leq b_i$, i.e.

$$^{a_i}X_i^{b_i} = \{(x_1,\ldots,x_n) \mid a_i \leq x_i \leq b_i\} \qquad (2)$$

If $a_i > b_i$ then $^{a_i}X_i^{b}$ is the empty set $\emptyset$.

Literals $^{a_i}X_i^{a_i}$, $a_i \in \{0,1,2,\ldots,h_i-1\}$, denoted briefly by
$X_i^{a_i}$, will be called <u>elementary literals</u>.

<u>Definition 2.</u> Any set of events L which can be represented as a product of literals, i.e.

$$L = \bigcap_{i \in I} X_i^{a_i, b_i}, \quad I \leqslant \{1, \ldots, n\} \tag{3}$$

will be called an <u>interval complex</u> (or simply <u>interval</u>).

We can easily see that interval complexes constitute in the space E, n-dimensional intervals, i.e. sets of all vectors which lie between some two arbitrary vectors, say, e^1 and e^2; e^1, $e^2 \in E$. Namely we would have $e^1 = (x_1^1, x_2^1, \ldots, x_n^1)$ and $e^2 = (x_1^2, x_2^2, \ldots, x_n^2)$, where $x_i^1 = a_i$ and $x_i^2 = b_i$ for $i \in I$, and $x_i^1 = 0$ and $x_i^2 = h_i - 1$ for $i \notin I$.

A product E_i of n elementary literals, i.e. $E_i = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ is an interval which consists of only one event, namely $e = (a_1, a_2, \ldots, a_n)$, thus $E_i = \{e\}$.

3. SET-THEORETIC OPERATIONS ON INTERVAL COMPLEXES

In this section we state a series of theorems which summarize the basic rules of set-theoretic operations on intervals. First, recall that union and intersection of sets are idempotent, commutative, associative and distributive; and the absorption laws hold. Next we have:

$$
\begin{aligned}
L \cup \emptyset &= L \\
L \wedge E &= L \quad \text{(identities)} \tag{4}
\end{aligned}
$$

$$
\begin{aligned}
L \cup E &= E \\
L \wedge \emptyset &= \emptyset \quad \text{(null elements)} \tag{5}
\end{aligned}
$$

$$
\begin{aligned}
L \cup \overline{L} &= E \\
L \wedge \overline{L} &= \emptyset \quad \text{(complements)} \tag{6}
\end{aligned}
$$

$$\overline{\overline{L}} = L \quad \text{(involution)} \tag{7}$$

$$
\begin{aligned}
\overline{L_1 \cup L_2} &= \overline{L}_1 \wedge \overline{L}_2 \\
\overline{L_1 L_2} &= \overline{L}_1 \cup \overline{L}_2 \quad \text{(de Morgan's Laws)} \tag{8}
\end{aligned}
$$

$$L_1 L_2 \cup \overline{L}_1 L_3 = L_1 L_2 \cup \overline{L}_1 L_3 \cup L_2 L_3 \quad \text{(consensus)} \tag{9}$$

Let $L_1 = \bigcap_{i \in I_1} {}^{a_i}X_i^{b_i}$, $\qquad L_2 = \bigcap_{i \in I_2} {}^{c_i}X_i^{d_i}$,

where $I_1, I_2 \subseteq \{1,2,\ldots,n\}$. We will assume that for $i \in \{1,2,\ldots,n\} \setminus I_1$, $a_i = 0$ and $b_i = h_i - 1$, and for $i \in \{1,2,\ldots,n\} \setminus I_2$, $c_i = 0$ and $d_i = h_i - 1$.

Theorem 1:

1. $L_1 \subseteq L_2$, iff $\forall i \in I_2$, $a_i \geq c_i$ and $b_i \leq d_i$

2. If $L_1 \subseteq L_2$ then $L_1(L_2 \vee L) = L_1$ and $L_2 \vee L_1 L = L_2$

Theorem 2:

$${}^{a}X_i^{b} \cup {}^{c}X_i^{d} = \begin{cases} {}^{\min(a,c)}X_i^{\max(b,d)} & \text{, if } \max(a,c) \leq \min(b,d) + 1 \\ \\ {}^{a}X_i^{b} \cup {}^{c}X_i^{d} & \text{, otherwise.} \end{cases}$$

The next theorem is a generalization of theorem 2.
Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$, $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$.

Theorem 3: If vector a is comparable with c (i.e. $a \geq c$ or $a \leq c$) and vector b is comparable with d, and $\max(a,c) \leq \min(b,d) + 1$, where $1 = (1,1,\ldots,1)$, then:

$$L_1 \vee L_2 = \bigcap_{i \in I_1 \vee I_2} {}^{\min(a_i,c_i)}X_i^{\max(b_i,d_i)}$$

If for every $i \in I_1 \vee I_2$, $\min(a_i, c_i) = 0$ and $\max(b_i, d_i) = h_i - 1$ then $L_1 \vee L_2 = E$.

Theorem 4:

$$\bigcup_{a=0}^{h_i - 1} (L \wedge X_i^a) = L$$

Proof:

$$\bigcup_{a=0}^{h_i - 1} (L \wedge X_i^a) = L \wedge (X_i^0 \cup X_i^1 \cup \ldots \cup X_i^{h_i - 1}) = L \wedge E = L. \quad \text{Q.E.D.}$$

More generally:

Theorem 5:

$$\bigcup_j (L \wedge {}^{a_{ij}}X_i^{b_{ij}}) = L$$

iff $\forall a \in \{0,1,\ldots,h_i-1\}$, $\exists j$ such that $a_{ij} \le a \le b_{ij}$

Theorem 6:

$$L_1 \wedge L_2 = \bigcap_{i \in I_1 \cup I_2} {}^{max(a_i,c_i)}X_i^{min(b_i,d_i)}$$

If for some i, $max(a_i,c_i) > min(b_i,d_i)$ then $L_1 \wedge L_2 = \emptyset$.

Theorem 7:

$$\overline{{}^{a_i}X_i^{b_i}} = {}^{0}X_i^{a_i-1} \cup {}^{b_i+1}X_i^{h_i-1}$$

Theorem 8:

1. $$\overline{\bigcap_{i \in I} {}^{a_i}X_i^{b_i}} = \bigcup_{i \in I} \overline{{}^{a_i}X_i^{b_i}}$$

2. $$\overline{\bigcup_{i \in I} {}^{a_i}X_i^{b_i}} = \bigcap_{i \in I} \overline{{}^{a_i}X_i^{b_i}}$$ (a special case of (8))

From theorems 2, 6 and 7 we see that the set L of all possible intervals in the space E ($\emptyset \in L$) is closed under $\wedge$ but not under $\cup$ and $-$. Since the operation $\wedge$ is associative, commutative and idempotent, the system with the carrier set L, binary operation $\wedge$, and nullary operations E and $\emptyset$. i.e.

$$< L, \wedge, E, \emptyset > \tag{10}$$

is a meet-semi-lattice.

Let us find the cardinality of L. Every element of L can be represented as a product of n literals. The number of different literals ${}^{a_i}X_i^{b_i}$ for a fixed $i \in \{1,2,\ldots,n\}$ is equal to the number of possible pairs (a_i,b_i), $a_i,b_i \{0,1,\ldots,h_i-1\}$ and $a_i \le b_i$, i.e. $\binom{h_i}{1} + \binom{h_i}{2} = \frac{h_i(h_i+1)}{2}$. Thus the number of intervals in L is

$$c(L) = \frac{1}{2^n} \prod_{i=1}^{n} h_i(h_i+1). \tag{11}$$

Let $L(E_i)$, $E_i \subseteq E$, denote the minimal interval under inclusion which contains E_i (i.e. the interval included in any other interval with such property). The unary operation

$$E_i \longrightarrow L(E_i)$$

of 2^E into itself has the following obvious properties:

1. $E_1 \subseteq E_2 \implies L(E_1) \subseteq L(E_2)$ (isotone)   (12)

2. $E_i \subseteq L(E_i)$ (extensive)   (13)

3. $L(L(E_i)) = L(E_i)$ (idempotent)   (14)

The above means that the operation $E \to L(E)$ is a closure operation on the poset $< 2^E, \subseteq >$, i.e. the set 2^E with inclusion $\subseteq$ as a partial ordering operation.

Let us define an operation $\sqcup$ on event sets E_i, called the normalized union, as:

$$\bigsqcup_i E_i = L\left(\bigcup_i E_i\right) \tag{15}$$

The set L is closed under the normalized union. Thus the system $<L, \wedge, \sqcup, E, \emptyset>$ is a lattice. Furthermore, it is a complete lattice as the intersection and normalized union of intervals from any subset of L also belong to L.

4. COVERS OF A MAPPING f

Assume that we are given two disjoint sets F^0 and F^1 of events from the space E. These sets define a mapping

$$f: E \longrightarrow \{1, 0, *\} \tag{16}$$

where $*$ denotes some unspecified value, and such that $F^0 = \{ e \mid f(e) = 0 \}$ and $F^1 = \{ e \mid f(e) = 1 \}$.

<u>Definition 3.</u> A set of intervals $D(f) = \{L_i\}_{i=1}^d$ is a <u>cover</u> of the mapping f if:

$$F^1 \subseteq \bigcup_{i=1}^d L_i \subseteq F^1 \cup F^* \tag{17}$$

where $F^* = \{ e \mid f(e) = * \}$.

Thus the cover $D(f)$ <u>distinguishes</u> the set F^1 from F^0 (we say $D(f)$ is a cover of F^1 against F^0). A first covering problem is how to provide a cover $D(f)$ with a minimum number of intervals. More generally, specifying a 'cost' functional for sets of intervals,

we can ask how a cover of minimal cost can be found. If
the space E is a space of binary vectors, then the latter
problem is parallel to the well known problem of finding the
minimal disjunctive normal form for an incompletely specified
switching function (where 'cost' is the number of literals,
in this case unprimed or primed variables). For the purpose
of the present paper, by the cost of a cover we will mean the
number of intervals in it.

Assume now that two given sets of events in E are
not disjoint, i.e. there exists a non-empty set of 'mixed'
events F^ϕ representing their intersection.

Formally, we will consider the extended mapping

$$f: \quad E \longrightarrow \{[1,0],*\} \tag{18}$$

and define:

$$F^1 = \{e \in E \;/\; f(e) = 1\} \tag{19}$$

$$F^0 = \{e \in E \;/\; f(e) = 0\} \tag{20}$$

$$F^\phi = \{e \in E \;/\; 0 < f(e) < 1\} \tag{21}$$

$$F^* = \{e \in E \;/\; f(e) = *\} = E \setminus (F^1 \cup F^0 \cup F^\phi). \tag{22}$$

Thus mixed events are here those events for which the mapping f
takes values properly between 0 and 1. A mixed event e can be
interpreted as having a nonzero conditional probability, rep-
resented by the value of f(e), of belonging to F^1. Events of F^ϕ
can be, on this basis, linearly ordered from those most F^1-like
to those least F^1-like (i.e. most F^0-like).

Assuming some threshold λ, where $\lambda \in [0,1]$, we define:

$$F^{1\lambda} = \{e \in E \;/\; \quad f(e) \geq \lambda\}$$

$$F^{0\lambda} = \{e \in E \;/\; \quad f(e) < \lambda\}$$

Sets $F^{1\lambda}$ and $F^{0\lambda}$ are disjoint sets, so we can now determine a cover
$F^{1\lambda}$ against $F^{0\lambda}$. To formalize the above idea we state

Definition 4. $D(f|\lambda) = \{L_i\}$ is a cover of f under λ if

$$F^{1\lambda} \subseteq \bigcup_i L_i \subseteq F^{1\lambda} \cup F^*. \tag{23}$$

If $\lambda = 1$ then $D(f|\lambda)$ reduces to $D(f)$.

It may turn out that two covers $D(f|\lambda_1)$ and $D(f|\lambda_2)$
can have considerably different cost, although values λ_1 and λ_2
are close.

In order to prevent this we partition the set $F^{\phi} = \{ e \mid 1 > f(e) > 0 \}$ into some arbitrary number p of classes of events $F^{\phi 1}, F^{\phi 2}, \ldots, F^{\phi p}$ (see fig. 1), defined as:

$$F^{\phi 1} = \{ e \mid \quad 1 \quad > f(e) > 1 - 1/p \}$$

$$F^{\phi 2} = \{ e \mid \quad 1 - 1/p \quad \geq f(e) > 1 - 2/p \}$$

$$\vdots \qquad\qquad \vdots$$

$$F^{\phi p} = \{ e \mid 1 - (p-1)/p \geq f(e) > \quad 0 \quad \}$$

The values of f for events in the same class can differ by not more than 1/p. Events in the same class will be considered as equivalent.

<u>Definition 5.</u> $D(f \mid \lambda, p) = \{L_i\}$ is a <u>free</u> <u>cover</u> of f <u>under</u> (λ, p) if

$$F^{1\lambda p} \subseteq \bigcup L_i \subseteq F^{1\lambda p} \cup F^{*} \qquad (24)$$

where

$$F^{1\lambda p} = F^{1\lambda} \setminus \Theta(F^{\phi j(\lambda)})$$

$\Theta(F^{\phi j(\lambda)})$ - a subset of $F^{\phi j(\lambda)}$

$j(\lambda)$ - a value j such that $1 - \dfrac{j-1}{p} \geq \lambda > 1 - j/p$.

The concept of the free cover $D(f \mid \lambda, p)$ allows us to cover only those events from the class $F^{\phi j(\lambda)}$ which can be covered with minimal increment of cost over the cost of covering the set $F^{1\lambda} \setminus F^{\phi j(\lambda)}$. To distinguish the covers $D(f \mid \lambda)$ from $D(f \mid \lambda, p)$, we will call the former <u>exact</u> <u>covers</u>.

The next concept to be introduced is that of a cover whose individual intervals cover events from $F^{1\lambda}$ in an ordered manner.

<u>Definition 6.</u> $\overrightarrow{D}(f \mid \lambda, p) = \langle L_1, L_2, \ldots, L_d \rangle$ is called an <u>ordered free</u> <u>cover</u> of f <u>under</u> (λ, p) if

$$D_k = \{ L_1, L_2, \ldots, L_k \}, \quad k = 1, 2, \ldots, d$$

are free covers of f under

$$\lambda_k = \min_{e \in E_k} f(e), \quad \text{where } E_k = L_1 \cup L_2 \cup \cdots \cup L_k$$

To distinguish the covers $D(f)$, $D(f \mid \lambda)$ or $D(f \mid \lambda, p)$ from the ordered covers $\overrightarrow{D}(f \mid \lambda, p)$, we will call the former <u>unordered covers</u>. We denote $\overrightarrow{D}(f \mid \lambda, 1)$ by $\overrightarrow{D}(f \mid \lambda)$ which, if $\lambda = 1$, reduces to $D(f)$.

Definition 7. An exact cover $D(f|\lambda)$, a free cover $D(f|\lambda,p)$ or an ordered free cover $\vec{D}(f|\lambda,p)$ which has the minimum number of intervals is called a minimal cover and denoted by $M(f|\lambda)$, $M(f|\lambda,p)$ or $\vec{M}(f|\lambda,p)$ respectively.

5. SYNTHESIS OF QUASI-MINIMAL COVERS

In this section we will briefly describe an application of the disjoint starts method [1,2,3,4], to the synthesis of the quasi-minimal covers.

5.1 The extension operation $\cup$

Definition 8. An extension operation $\cup$ on event set E_1 relative to event set E_2 is defined as:

$$E_1 \cup E_2 = \bigcup_{L_i \in \Lambda} L_i$$

where $\Lambda = \{ L_i \in L \ / \ E_1 \cap L_i \neq \emptyset \text{ and } L_i \subset E_2 \}$.

If Λ is empty we assign $E_1 \cup E_2 = \emptyset$.

The extension operation satisfies the following (asymmetrical) distributive properties:

Theorem 9: Given event sets E, E_i ($i = 1,2,\ldots.$) we have:

(1) $\left(\bigcup_i E_i \right) \cup E = \bigcup_i (E_i \cup E)$

(2) $E \cup \left(\bigcap_i E_i \right) = \bigcap_i (E \cup E_i).$

Proof is given in [5].

Theorem 10:

$$E_1 \cup E_2 = \bigcup_{e_j \in E_1} \bigcap_{e_i \in \overline{E}_2} (\{e_j\} \cup \overline{\{e_i\}})$$

where the order of the union and intersection is irrelevant.

Proof is given in [5].

5.2 Definition of a star $G(e|\lambda)$ and an algorithm for its generation

Definition 9. L is a maximal interval in f under λ if

$$L \subseteq F^{1\lambda} \cup F^* \tag{25}$$

and if it is maximal under inclusion. Maximal intervals in f under λ are denoted by L_i^λ, $i = 1,2,\ldots$

Definition 10. The interval star $G(e|\lambda)$ under λ of an event $e \in F^{1\lambda}$ is a set of all maximal intervals under λ which cover the event e, i.e.

$$G(e|\lambda) = \{ L_i^\lambda \ | \ L_i^\lambda \ni e \} \tag{26}$$

We denote $G(e|\lambda)$ by $G(e)$. The union of intervals from $G(e|\lambda)$ is denoted by $G^\cup(e|\lambda)$.

Theorem 11:
$$G^{U}(e|\lambda) = \bigcap_{e_i \in F^{0\lambda}} (\{e\} \cup \{\overline{e_i}\})$$

Proof:
$$G^{U}(e|\lambda) = \{e\} \cup \overline{F^{0\lambda}} = \{e\} \cup \bigcap_{e_i \in F^{0\lambda}} \{\overline{e_i}\}$$

Applying now theorem 9 part (2) we complete the proof.   Q.E.D

The following algorithm for the generation of a star $G(e|\lambda)$ follows from theorem 11.

Given $e \in F^{1\lambda}$ and $F^{0\lambda} = \{e_j\}_{i=1}^{z}$.

1. Determine for $i = 1,2,\ldots,z$
    $\{\overline{e_i}\}$ and then form $D_i = \{e\} \cup \{\overline{e_i}\}$

2. Set up the function $\bigcap_{i=1}^{z} D_i$ and find its irredundant
    expression by multiplying each term D_i by all the others
    and applying the absorption laws.

3. $G(e|\lambda)$ is the set of terms of the expression thus
    obtained.

5.3 Synthesis of a cover $M^q (f|\lambda)$ by algorithm A^q

The problem of synthesis of a minimal cover $M(f,\lambda)$ is a
particular case of the general covering problem described in [2]. As
indicated in this paper even in case of a relatively simple covering
problem the number of operations required for its exact solution may
not be feasible even with the fastest computers. Consequently, the
most desirable are good methods for an approximate solution which
allow to drastically reduce the number of operations but also give
some measure of distance to the minimum.

The previously mentioned method of disjoined stars, when
realized by locally optimal decisions, generates—with a relatively
small number of operations and memory requirements—a so-called quasi-
minimal cover, which is either minimal or approximately minimal.
Furthermore, when we cannot state that obtained solution is minimal,
it provides an estimate of the maximal possible distance between the
obtained solution and the minimal one.

The fundamental theorem from which this method stems
(expressed in our terms) is following.

Let us assume that we are given (by realizing some algorithm)
a family of stars $G^r = \{G_i(e|\lambda)\}$, $e \in F^{1\lambda}$, such that any two stars
chosen from it are disjoint sets (we say G^r is a family of disjoint
stars).

Theorem 12: The number of intervals in the minimal cover $M(f|\lambda)$ satisfies the relation:

$$c(M(f|\lambda)) \geq c(G^r)$$

The theorem implies that if we have a cover $D(f|\lambda)$ and know the number $\varrho = c(G^r)$, then the difference $\Delta = (D(f|\lambda)) - \varrho$ can be viewed as an estimate of the difference between the number of elements in this cover, and in a minimal one. If we can next find another family of disjoint stars with a greater number of elements, say ϱ_1, then we can improve our estimate, namely:

$$\Delta_1 = (D(f|\lambda)) - \varrho_1 < \Delta$$

If on the other hand we can find another cover with a smaller number of intervals, then obviously our estimate will also improve, and it may turn out then Δ will become 0. This will mean that we found the minimal solution.

A possible algorithm for accomplishing the above ideas with a view toward a solution of the generally stated covering problem, was described in paper [2] (algorithm A^Q). In the formulation given there it was assumed that a cover consists of some sets (not concretely specified), called complexes. In our case the complexes are specified as intervals (interval complexes). Thus in order to apply the algorithm A^q for our purpose we only need to make use of the algorithm for generating a star $G(f|\lambda)$, described in section [5.2].

The flow diagram of algorithm A^q in the form adapted to our notation is shown in fig. 2.

The sign := denotes assignment statement (as in Algol 60). F^p is an auxiliary variable. $O(F^p; e_1)$ denotes the operation of choosing the event with the smallest number from the set specified by the current value of F^p and assigning the notation e_1 to that event. L^q denotes an interval in $G(e_1|\lambda)$, called a _quasi-extremal_, which covers the maximum number of events in the set constituting the current value of variable $F^{1\lambda}$. It can be noticed that this is a locally optimal decision about the choice of an interval from the given star (generally not unique). The last value of M^q constitutes our solution, the quasi-minimal cover $M^q(f|\lambda)$. Value Δ is an estimate of the maximal possible difference between the cover $M^q(f|\lambda)$ and a minimal cover $M(f|\lambda)$, expressed in number intervals, i.e.

$$c(M^q(f|\lambda)) - c(M(f|\lambda)) \leq \Delta . \tag{27}$$

If after the first execution of the algorithm, Δ is considered to be too large, the better estimate (and/or solution) may be obtained realizing next iterations, e.g. in the way described in [3].

To synthesize a quasi-minimal free cover $M^q(f,\lambda,p)$ using this algorithm, we substitute in the flow diagram (fig. 2) the set $F^{1\lambda}$ by $F^{1\lambda} \setminus F^{\phi i(\lambda)}$, and after determining the cover, one additional quasi-extremal from a star $G(e|\lambda)$, $e \in F^{\phi i(\lambda)}$ is determined and added

to the solution. An extension of the algorithm for synthesis quasi-minimal ordered covers is described in [5].

6. REMARKS ON APPLICATIONS

A cover of a mapping f consisting of multidimensional intervals can be interpreted as a set of 'filters' for recognizing events from a signal class (represented by $F^{1\lambda}$) for pattern recognition and picture processing purposes, e.g. for discriminating regions of different textures, striping of background, local feature extraction, border detection, etc.

In case of an ordered cover the individual filters correspond to the consecutive points on the optimum receiver-operating-characteristic (ROC), defined as in statistical decision theory. The above ROC curve can be obtained by optimal ordering events from the two classes to be distinguished, with regard to the likelihood ratio of their frequency occurence.

7. REFERENCES

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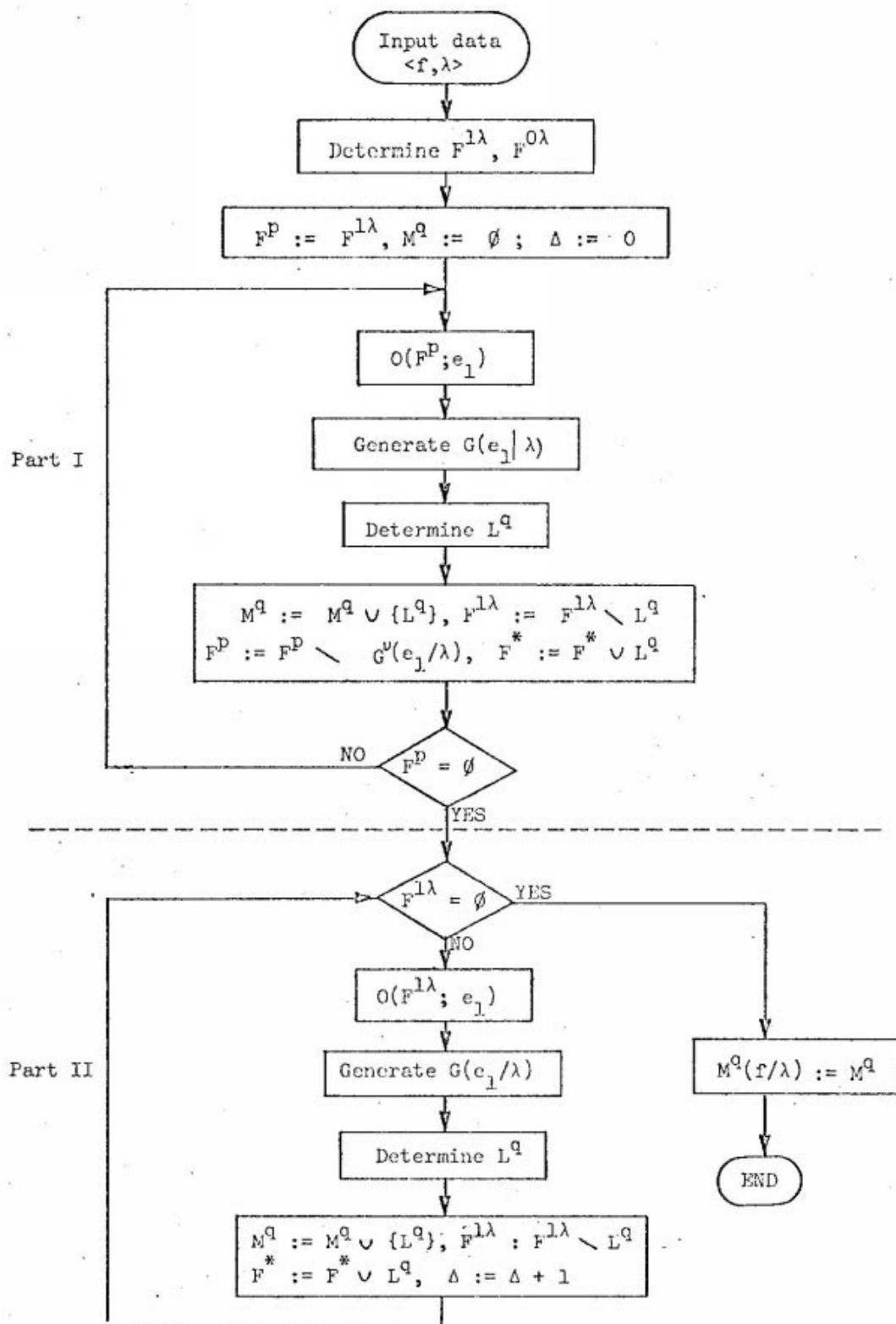
Input data
$\langle f, \lambda \rangle$

Determine $F^{1\lambda}$, $F^{0\lambda}$

$F^p := F^{1\lambda}$, $M^q := \emptyset$; $\Delta := 0$

Part I

$O(F^p; e_1)$

Generate $G(e_1 | \lambda)$

Determine L^q

$M^q := M^q \cup \{L^q\}$, $F^{1\lambda} := F^{1\lambda} \setminus L^q$
$F^p := F^p \setminus G^\cup(e_1/\lambda)$, $F^* := F^* \cup L^q$

$F^p = \emptyset$ NO YES

$F^{1\lambda} = \emptyset$ YES NO

Part II

$O(F^{1\lambda}; e_1)$

Generate $G(e_1/\lambda)$

Determine L^q

$M^q := M^q \cup \{L^q\}$, $F^{1\lambda} : F^{1\lambda} \setminus L^q$
$F^* := F^* \cup L^q$, $\Delta := \Delta + 1$

$M^q(f/\lambda) := M^q$

END

Fig. 2