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INTERVAL GENERALIZATION OF SWITCHING THEORY

by

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ABSTRACT

The paper considers:

(1) a Boolean algebra $< 2^E, \cup, \cap, -, E, \emptyset >$ of event sets E_i from a discrete finite vector space E, and

(2) mappings f from the set E into {[0,1], *}, where * represents some unspecified value. A special case of the above is the Boolean algebra and Boolean functions considered in switching theory, where E is a space of binary vectors and f maps E into {0,1,*}, i.e. into the endpoints of the interval [0,1] and *.

A meet semi-lattice of multidimensional intervals (interval complexes) in E is introduced and then the concepts of exact, free, unordered and ordered interval covers of f are defined.

The simplest case of a cover - an unordered exact cover of a set $F^{1\lambda}$ against $F^{0\lambda}$ - is defined as a set of interval complexes whose set-theoretic union covers a given subset $F^{1\lambda} \subseteq E$ (defined as $\{ e \mid f(e) \geq \lambda \}$) and does not cover any element of another given subset $F^{0\lambda} \subseteq E \setminus F^1$.

The concept of ordered covers was developed to accomodate a preferential order in covering the set of 'mixed' events, defined as $\{ e \mid 0 < f(e) < 1 \}$ (a case not considered in classical switching theory).

The synthesis algorithm of covers is based on the 'method of disjoint stars', which has proved to be very useful for synthesis of complex switching systems. Quasi-minimal covers, produced by this method, are either minimal or approximately minimal. However, when we cannot state that the obtained solution is minimal, an estimate of its maximal possible distance to the minimum is provided.

Applications of the interval coverings concepts to pattern recognition and picture filtering are delineated.

1. INTRODUCTION

Some concepts and methods, initially developed for switching theory purposes, seem to have more universal application, if properly generalized. The generalization described in this paper stems from three observations:

(1) The coverings arising in the minimization of switching circuits can be viewed as a limiting case of interval coverings, introduced in the paper.

(2) The so-called method of disjoint stars, which was originally developed to provide a minimal or quasi-minimal solution of the covering problem in switching theory [1,3,4], and then extended to provide the quasi-minimal solution of the general covering problem [2], can be applied in particular to the synthesis of the above-mentioned interval coverings.

(3) The concept of a covering can be extended in yet another direction to accommodate a preferential ordering of the elements to be covered.

A need for the generalization described in the paper first appeared when we were considering some problems of pattern recognition and signal detection theory.

2. NOTATION AND DEFINITION OF AN INTERVAL COMPLEX

? set

Elements of a discrete finite vector space $\mathbf{E}$ will be referred to as events $e^j = (x_1,\ldots,x_n)$. Components x_i take their values from the sets $\{0,1,2,\ldots,h_i-1\}$, $i = 1,2,\ldots,n$. We will assume that the index j is given by

$$j = x_n + \sum_{i=1}^{n-1}\left(x_{n-i}\prod_{k=0}^{i-1} h_{n-k}\right) ; \tag{1}$$

thus, as can be verified, j uniquely determines a vector e^j. The value j will be called the <u>number</u> of the <u>event</u> e^j. When an event $e \epsilon \mathbf{E}$ is given, its number will be denoted by $\gamma(e)$. For example, if $e = (2,2,2,1)$, assuming that $h_1 = 5$, $h_2 = 3$, $h_3 = 3$ and $h_4 = 2$, $\gamma\,(e) = 1 + 2\,(2) + 2\,(2\cdot 3) + 2\,(2\cdot 3\cdot 3) = 53$.

$E_1 \cup E_2$ and $E_1 \cap E_2$ (or E_1E_2) will denote the set-theoretic union and intersection of E_1 and E_2 respectively, where E_1 and E_2 are sets of events. $\overline{E}_i$ is the complement of an event set E_i, defined as $\mathbf{E} \setminus E_i$, where $\setminus$ is the set-theoretic subtraction. The cardinality of a set L will be denoted by $c(L)$.

<u>Definition 1.</u> By ${}^{a_i}X_i^{b_i}$, called a <u>literal</u>, we will denote the set of all events $(x_1,\ldots,x_n)$ from the space $\mathbf{E}$ such that $a_i \leq x_i \leq b_i$, i.e.

$${}^{a_i}X_i^{b_i} = \{(x_1,\ldots,x_n) \mid a_i \leq x_i \leq b_i\} \tag{2}$$

If $a_i > b_i$ then ${}^{a_i}X_i^{b_i}$ is the empty set $\emptyset$.

Literals ${}^{a_i}X_i^{a_i}$, $a_i \epsilon \{0,1,2,\ldots,h_i-1\}$, denoted briefly by $X_i^{a_i}$, will be called <u>elementary literals</u>.

Definition 2. Any set of events L which can be represented as a product of literals, i.e.

$$L = \bigcap_{i \in I} X_i^{a_i b_i}, \quad I \subseteq \{1,\ldots,n\} \tag{3}$$

will be called an interval complex (or simply interval).

We can easily see that interval complexes constitute in the space E, n-dimentional intervals, i.e. sets of all vectors which lie between some two arbitrary vectors, say, e^1 and e^2, $e^1, e^2 \in E$. Namely we would have $e^1 = (x_1^1, x_2^1, \ldots, x_n^1)$ and $e^2 = (x_1^2, x_2^2, \ldots, x_n^2)$, where $x_i^1 = a_i$ and $x_i^2 = b_i$ for $i \in I$, and $x_i^1 = 0$ and $x_i^2 = h_i - 1$ for $i \notin I$.

A product E_i of n elementary literals, i.e. $E_i = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ is an interval which consists of only one event, namely $e = (a_1, a_2, \ldots, a_n)$, thus $E_i = \{e\}$.

3. SET-THEORETIC OPERATIONS ON INTERVAL COMPLEXES

In this section we state a series of theorems which summarize the basic rules of set-theoretic operations on intervals. First, recall that union and intersection of sets are idempotent, commutative, associative and distributive; and the absorption laws hold. Next we have:

$$L \cup \emptyset = L$$
$$L \cap E = L \qquad \text{(identities)} \tag{4}$$

$$L \cup E = E$$
$$L \cap \emptyset = \emptyset \qquad \text{(null elements)} \tag{5}$$

$$L \cup \overline{L} = E$$
$$L \cap \overline{L} = \emptyset \qquad \text{(complements)} \tag{6}$$

$$\overline{\overline{L}} = L \qquad \text{(involution)} \tag{7}$$

$$\overline{L_1 \cup L_2} = \overline{L}_1 \cap \overline{L}_2$$
$$\overline{L_1 L_2} = \overline{L}_1 \cup \overline{L}_2 \qquad \text{(de Morgan's Laws)} \tag{8}$$

$$L_1 L_2 \cup \overline{L}_1 L_3 = L_1 L_2 \cup \overline{L}_1 L_3 \cup L_2 L_3 \qquad \text{(consensus)} \tag{9}$$

Let $L_1 = \bigcap_{i \in I_1} {}^{a_i}X_i^{b_i}$, $\quad L_2 = \bigcap_{i \in I_2} {}^{c_i}X_i^{d_i}$,

where $I_1, I_2 \subseteq \{1,2,\ldots,n\}$. We will assume that for $i \in \{1,2,\ldots,n\} \setminus I_1$, $a_i = 0$ and $b_i = h_i - 1$, and for $i \in \{1,2,\ldots,n\} \setminus I_2$, $c_i = 0$ and $d_i = h_i - 1$.

<u>Theorem 1:</u>

1. $L_1 \leq L_2$, iff $\forall i \in I_2$, $a_i \geq c_i$ and $b_i \leq d_i$
2. If $L_1 \leq L_2$ then $L_1(L_2 \cup L) = L_1$ and $L_2 \vee L_1 L = L_2$

<u>Theorem 2:</u>

$$
{}^{a}X_i^{b} \cup {}^{c}X_i^{d} = \begin{cases} {}^{\min(a,c)}X_i^{\max(b,d)} & \text{, if } \max(a,c) \leq \min(b,d) + 1 \\ {}^{a}X_i^{b} \cup {}^{c}X_i^{d} & \text{, otherwise.} \end{cases}
$$

The next theorem is a generalization of theorem 2.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.

<u>Theorem 3:</u> If vector $\mathbf{a}$ is comparable with $\mathbf{c}$ (i.e. $\mathbf{a} \geq \mathbf{c}$ or $\mathbf{a} \leq \mathbf{c}$) and vector $\mathbf{b}$ is comparable with $\mathbf{d}$, and $\max(\mathbf{a},\mathbf{c}) \leq \min(\mathbf{b},\mathbf{d}) + \mathbf{1}$, where $\mathbf{1} = (1,1,\ldots,1)$, then:

$$
L_1 \cup L_2 = \bigcap_{i \in I_1 \cup I_2} {}^{\min(a_i,c_i)}X_i^{\max(b_i,d_i)}
$$

If for every $i \in I_1 \vee I_2$, $\min(a_i, c_i) = 0$ and $\max(b_i, d_i) = h_i - 1$ then $L_1 \vee L_2 = E$.

<u>Theorem 4:</u>

$$
\bigcup_{a=0}^{h_i-1} (L \cap X_i^{a}) = L
$$

<u>Proof:</u>

$$
\bigcup_{a=0}^{h_i-1} (L \cap X_i^{a}) = L \cap (X_i^{0} \vee X_i^{1} \vee \ldots \vee X_i^{h_i-1}) = L \cap E = L. \quad \text{Q.E.D.}
$$

More generally:

Theorem 5:

$$\bigcup_j (L \cap {}^{a_{ij}}X_i^{b_{ij}}) = L$$

iff $\forall a \in \{0,1,\ldots,n_i-1\}$, $\exists j$ such that $a_{ij} \leq a \leq b_{ij}$

Theorem 6:

$$L_1 \cap L_2 = \bigcap_{i \in I_1 \cup I_2} {}^{\max(a_i,c_i)}X_i^{\min(b_i,d_i)}$$

If for some i, $\max(a_i,c_i) > \min(b_i,d_i)$ then $L_1 \cap L_2 = \emptyset$.

Theorem 7:

$$\overline{{}^{a_i}X_i^{b_i}} = {}^{0}X_i^{a_i-1} \cup {}^{b_i+1}X_i^{h_i-1}$$

Theorem 8:

1. $$\overline{\bigcap_{i \in I} {}^{a_i}X_i^{b_i}} = \bigcup_{i \in I} \overline{{}^{a_i}X_i^{b_i}}$$

2. $$\overline{\bigcup_{i \in I} {}^{a_i}X_i^{b_i}} = \bigcap_{i \in I} \overline{{}^{a_i}X_i^{b_i}}$$ (a special case of (8))

From theorems 2, 6 and 7 we see that the set L of all possible intervals in the space E ($\emptyset \in L$) is closed under $\cap$ but not under $\cup$ and $-$. Since the operation $\cap$ is associative, commutative and idempotent, the system with the carrier set L, binary operation $\cap$, and nullary operations E and $\emptyset$, i.e.

$$< L, \cap, E, \emptyset > \qquad (10)$$

is a meet semi-lattice.

Let us find the cardinality of L. Every element of L can be represented as a product of n literals. The number of different literals ${}^{a_i}X_i^{b_i}$ for a fixed $i \in \{1,2,\ldots,n\}$ is equal to the number of possible pairs (a_i,b_i), a_i,b_i $\{0,1,\ldots,h_i-1\}$ and $a_i \leq b_i$, i.e. $\binom{h_i}{1} + \binom{h_i}{2} = \frac{h_i(h_i+1)}{2}$. Thus the number of intervals in L is

$$c(L) = \frac{1}{2^n} \prod_{i=1}^{n} h_i(h_i+1). \qquad (11)$$

Let $L(E_i)$, $E_i \subseteq E$, denote the minimal interval under inclusion which contains E_i (i.e. the interval included in any other interval with such property). The unary operation

$$E_i \longrightarrow L(E_i)$$

of 2^E into itself has the following obvious properties:

1. $E_1 \subseteq E_2 \implies L(E_1) \subseteq L(E_2)$ (isotone) (12)

2. $E_i \subseteq L(E_i)$ (extensive) (13)

3. $L(L(E_i)) = L(E_i)$ (idempotent) (14)

The above means that the operation $E \longrightarrow L(E)$ is a closure operation on the poset $< 2^E, \subseteq >$, i.e. the set 2^E with inclusion $\subseteq$ as a partial ordering operation.

Let us define an operation $\sqcup$ on event sets E_i, called the normalized union, as:

$$\bigsqcup_i E_i = L(\bigcup_i E_i) \tag{15}$$

The set L is closed under the normalized union. Thus the system $<L, \cap, \sqcup, E, \emptyset>$ is a lattice. Furthermore, it is a complete lattice as the intersection and normalized union of intervals from any subset of L also belong to L.

4. COVERS OF A MAPPING f

Assume that we are given two disjoint sets F^0 and F^1 of events from the space E. These sets define a mapping

$$f: E \longrightarrow \{1, 0, *\} \tag{16}$$

where * denotes some unspecified value, and such that $F^0 = \{ e \mid f(e) = 0 \}$ and $F^1 = \{ e \mid f(e) = 1 \}$.

Definition 3. A set of intervals $D(f) = \{L_i\}_{i=1}^{d}$ is a cover of the mapping f if:

$$F^1 \subseteq \bigcup_{i=1}^{d} L_i \subseteq F^1 \cup F^* \tag{17}$$

where $F^* = \{ e \mid f(e) = * \}$.

Thus the cover $D(f)$ distinguishes the set F^1 from F^0 (we also say $D(f)$ is a cover of F^1 against F^0). A first covering problem is how to provide a cover $D(f)$ with a minimum number of intervals. More generally, specifying a 'cost' functional for sets of intervals, we can ask how a cover of minimal cost can be found. If the space E is a space of binary vectors, then the latter problem is parallel to the well known problem of finding the minimal disjunctive normal form for an incompletely specified switching function (where 'cost' is the number of literals, in this case unprimed or primed variables). For the purpose of the present paper, by the <u>cost</u> of a <u>cover</u> we will mean the number of intervals in it.

Assume now that two given sets of events in E are not disjoint, i.e. there exists a non-empty set of 'mixed' events F^{ϕ} representing their intersection.

Formally, we will consider the extended mapping

$$f: \ E \to \{[1,0],*\} \tag{18}$$

and define:

$$F^1 = \{e \in E \ / \ f(e) = 1\} \tag{19}$$

$$F^0 = \{e \in E \ / \ f(e) = 0\} \tag{20}$$

$$F^{\phi} = \{e \in E \ / \ 0 < f(e) < 1\} \tag{21}$$

$$F^* = \{e \in E \ / \ f(e) = *\} = E \setminus (F^1 \cup F^0 \cup F^{\phi}). \tag{22}$$

Thus mixed events are here those events for which the mapping f takes values properly between 0 and 1. A mixed event e can be interpreted as having a nonzero conditional probability, represented by the value of f(e), of belonging to F^1. Events of F^{ϕ} can be, on this basis, linearly ordered from those most F^1-like to those least F^1-like (i.e. most F^0-like).

Assuming some threshold λ, where $\lambda \in [0,1]$, we define:

$$F^{1\lambda} = \{e \in E \ / \quad f(e) \geq \lambda\}$$

$$F^{0\lambda} = \{e \in E \ / \quad f(e) < \lambda\}$$

Sets $F^{1\lambda}$ and $F^{0\lambda}$ are disjoint sets, so we can now determine a cover $F^{1\lambda}$ against $F^{0\lambda}$. To formalize the above idea we state

<u>Definition 4.</u> $D(f|\lambda) = \{L_i\}$ is a <u>cover</u> of f <u>under</u> λ if

$$F^{1\lambda} \subseteq \bigcup_i L_i \subseteq F^{1\lambda} \cup F^* . \tag{23}$$

If $\lambda = 1$ then $D(f|\lambda)$ reduces to $D(f)$, defined in (17).

It may turn out that two covers $D(f|\lambda_1)$ and $D(f|\lambda_2)$ can have considerably different cost, although values λ_1 and λ_2 are close.

In order to prevent this we partition the set $F^{\phi} = \{ e \mid 1 > f(e) > 0 \}$ into some arbitrary number p of classes of events $F^{\phi 1}, F^{\phi 2}, \ldots, F^{\phi p}$ (see fig. 1), defined as:

$$\begin{aligned}
F^{\phi 1} &= \{ e \mid \quad 1 \quad > f(e) > 1 - 1/p \} \\
F^{\phi 2} &= \{ e \mid \quad 1 - 1/p \quad \geq f(e) > 1 - 2/p \} \\
&\vdots \\
F^{\phi p} &= \{ e \mid 1 - (p-1)/p \geq f(e) > \quad 0 \quad \}
\end{aligned}$$

The values of f for events in the same class can differ by not more than $1/p$. Events in the same class will be considered as equivalent.

<u>Definition 5.</u> $D(f|\lambda,p) = \{L_i\}$ is a <u>free</u> <u>cover</u> of f <u>under</u> (λ,p) if

$$F^{1\lambda p} \subseteq \bigcup L_i \subseteq F^{1\lambda p} \cup F^{*} \tag{24}$$

where $\quad F^{1\lambda p} = F^{1\lambda} \setminus \overline{\Theta}(F^{\phi j(\lambda)})$

$\Theta(F^{\phi j(\lambda)})$ - a subset of $F^{\phi j(\lambda)}$

$j(\lambda)$ - a value j such that $1 - \dfrac{j-1}{p} \geq \lambda > 1 - j/p$.

The concept of the free cover $D(f|\lambda,p)$ allows us to cover only those events from the class $F^{\phi j(\lambda)}$ which can be covered with minimal increment of cost over the cost of covering the set $F^{1\lambda} \setminus F^{\phi j(\lambda)}$. To distinguish the covers $D(f|\lambda)$ from $D(f|\lambda,p)$, we will call the former <u>exact</u> <u>covers</u>.

The next concept to be introduced is that of a cover whose individual intervals cover events from $F^{1\lambda}$ in an ordered manner.

<u>Definition 6.</u> $\overrightarrow{D}(f|\lambda,p) = \langle L_1, L_2, \ldots, L_d \rangle$ is called an <u>ordered</u> <u>free</u> <u>cover</u> of f <u>under</u> (λ,p) if

$$D_k = \{ L_1, L_2, \ldots, L_k \}, \quad k = 1,2,\ldots,d$$

are free covers of f under

$$\lambda_k = \min_{e \in E_k} f(e) \;, \quad \text{where } E_k = L_1 \cup L_2 \cup \ldots \cup L_k$$

To distinguish the covers $D(f)$, $D(f|\lambda)$ or $D(f|\lambda,p)$ from the ordered covers $\overrightarrow{D}(f|\lambda,p)$, we will call the former <u>unordered</u> <u>covers</u>. We denote $\overrightarrow{D}(f|\lambda,1)$ by $\overrightarrow{D}(f|\lambda)$ which, if $\lambda = 1$, reduces to $D(f)$.

Let $\{E_1, E_2, \ldots, E_t\}$ be a family of event sets and E an event set. We adopt the following notation:

$$\{E_1, E_2, \ldots, E_t\}^{\nu} = \bigcup_{i=1}^{t} E_i \tag{25}$$

$$\phi^{\nu} = \phi \tag{26}$$

$$E^{1/\nu} = \sqrt[\nu]{E} = L_E \tag{27}$$

where $L_E = \{L_1, L_2, \ldots\}$ is the set of all intervals which are maximal under inclusion[1] with regard to condition $L_k \subseteq E$, $k = 1,2,\ldots$ (briefly, maximal intervals included in E).

Thus: $$\sqrt[\nu]{E} = \{\max L_k \mid L_k \subseteq E\} \tag{28}$$

It can easily been seen that:

$$L_E^{\nu} = \{L_1, L_2, \ldots\}^{\nu} = E \tag{29}$$

and

$$\sqrt[\nu]{L_E^{\nu}} = L_E \tag{30}$$

According to the introduced notation, the set $\sqrt[\nu]{F^{1\lambda}}$ is a (usually redundant) cover of $F^{1\lambda}$ against $F^{0\lambda}$ - if $F^* = \emptyset$, or, if $F^* \neq \emptyset$ - a cover $F^{1\lambda}$ against $E \setminus F^{1\lambda} = F^{0\lambda} \cup F^*$.

Definition 7. L is called a maximal interval in f under λ if it is maximal under inclusion with regard to the condition:

$$L \subseteq F^{1\lambda} \cup F^* \tag{31}$$

Maximal intervals in f under λ are denoted by L_k^{λ}, $k = 1,2,\ldots,$. It can easily be seen that if E is a space

---

1) A set S satisfying condition p is maximal (minimal) under inclusion with regard to p, denoted by $S = \max S/p$ $(S = \min S/p)$, if there does not exist a superset (subset) of it also satisfying condition p. In general, there can be many maximal (minimal) under inclusion sets satisfying certain condition p. Family of such sets, i.e. $\{S_i / S_i = \max S_i/p\}$ is denoted briefly by $\{\max S_i/p\}$.

of binary vectors and the set of 'mixed' events F^{ϕ} is empty, then the maximal intervals L_k^1, (i.e. the maximal intervals in f under $\lambda = 1$) correspond to prime implicants of an incompletely specified switching function $f: E \to \{1,0,*\}$.

Definition 8. An exact cover $D(f|\lambda)$ or a free cover $D(f|\lambda,p)$ is called an irredundant cover if it consists of maximal intervals L_k^{λ} and if it is minimal under inclusion.

Definition 8 implies that deleting any interval from an irredundant cover $D(f|\lambda)$ or $D(f|\lambda,p)$ will cause them to no longer be covers. An irredundant cover $D(f|\lambda)$ or $D(f|\lambda,p)$ may be obtained from the set

$$\sqrt[\cup]{F^{1\lambda} \cup F^{*}}$$

by removing from it a maximal under inclusion subset of intervals such that the union of intervals in the remainder still covers set $F^{1\lambda}$ or $F^{1\lambda p}$, respectively.

In general there can be very many different irredundant covers $D(f|\lambda)$ or $D(f|\lambda,p)$. It is easy to see that if E is the space of binary vectors and $F^{\phi} = \emptyset$, an irredundant cover $D(f|1)$ corresponds to an irredundant disjunctive normal expression of a switching function $f: E \to \{1,0,*\}$.

Definition 9.
The minimal exact cover $M(f|\lambda)$, minimal free cover $M(f|\lambda,p)$ and minimal free ordered cover $M(f|\lambda,p)$ is a cover $D(f|\lambda)$, $D(f|\lambda,p)$ and $\overline{D(f|\lambda,p)}$ - respectively, which has a minimum number of intervals.

5. SYNTHESIS OF QUASI-MINIMAL COVERS

In this section we will briefly describe an application of the disjoint stars method [1,2,3,4] to the synthesis of the quasi-minimal covers.

5.1 The extension operation $\curlyvee$

Definition 10. An extension operation $\curlyvee$ on event set E_1 relative to event set E_2 is defined as:

$$E_1 \curlyvee E_2 = \Lambda^{\cup} \qquad (32)$$

where $\Lambda = \{L_k \in \sqrt[\cup]{E_2} \mid L_k \cap E_1 \neq \emptyset\}$.
According to (26), if $\Lambda = \emptyset$ then $E_1 \curlyvee E_2 = \emptyset$.

Since the union of intervals from any subset of $\sqrt[\vee]{E_2}$ is also included in E_2 we can state the following:

<u>Theorem 9:</u>

$$(1)\quad E_1 \veebar E_2 \subseteq E_2, \text{ if } E_1 \cap E_2 \neq \emptyset$$

$$(2)\quad E_1 \veebar E_2 = \emptyset, \text{ otherwise}$$

If E_2 is an interval then we have the stronger

<u>Theorem 10:</u>

$$E \veebar L = \begin{cases} L, & \text{if } E \cap L \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

where E is an event set and L an interval.

The extension operation satisfies the following asymetrical distributive properties:

<u>Theorem 11:</u>

$$\text{(a)}\quad E \veebar \left(\bigcap_i E_i\right) = \bigcap_i (E \veebar E_i)$$

$$\text{(b)}\quad \left(\bigcup_i E_i\right) \veebar E = \bigcup_i (E_i \veebar E)$$

<u>Proof:</u>

Part (a). If E_i are intervals then assertion follows immediately from the fact that any product of intervals is also an interval and from theorem 10.

We denote $E \veebar \left(\bigcap_i E_i\right) = \Lambda^{\vee}$ (i)

and $\bigcap_i (E \veebar E_i) = \bigcap_i \Lambda_i^{\vee}$ (ii)

According to definition 10 and (27):

$$\Lambda = \{L_k \in \sqrt[\vee]{\bigcap_i E_i} \mid L_k \cap E \neq \emptyset\} =$$

$$= \{\max L_k \mid L_k \cap E \neq \emptyset \text{ and } L_k \subseteq \bigcap_i E_i\} \qquad \text{(iii)}$$

Theorem

$E_1 \veebar (L_1 \cup L_2 \cup \ \ ) = \bigcup (E \veebar L_i^*)$

where $L_i^* \in \sqrt{L_1 \cup L_2 \cup \ldots}$

$$\Lambda_i = \{ L_k \in \sqrt[\vee]{E_i} \mid L_k \cap E \neq \emptyset \} =$$

$$\{ \max L_k \mid L_k \cap E \neq \emptyset \text{ and } L_k \leq E_i \} \qquad \text{(iv)}$$

Consider (i) and (iii) : $L_a \leq \bigcap_i E_i$ implies that $L_a \leq E_i$ for every E_i. So if $L_a \in \Lambda$ then also $L_a \in \Lambda_i$, for every E_i. Thus $\Lambda \leq \bigcap_i \Lambda_i$, which implies that $\check{\Lambda} \leq \bigcap_i \check{\Lambda}_i$.

Consider now (ii) and (iv). Set $\bigcap_i \check{\Lambda}_i$ can be uniquely represented by set of intervals $\sqrt[\vee]{\bigcap_i \check{\Lambda}_i}$. Let $L_a \in \sqrt[\vee]{\bigcap_i \check{\Lambda}_i}$ and $L_a \cap E \neq \emptyset$. From (28), $L_a \in \{ \max L_k \mid L_k \leq \bigcap_i \check{\Lambda}_i \}$ and then $L_a \leq \check{\Lambda}_i$ for every Λ_i. From that and (iv) we have $L_a \leq E_i$ for every E_i. It implies that $L_a \leq \bigcap_i E_i$, and finally, according to (iii), $L_a \in \Lambda$.

Therefore $\sqrt[\vee]{\bigcap_i \check{\Lambda}_i} \leq \Lambda$, what implies $\bigcap_i \check{\Lambda}_i \leq \check{\Lambda}$.

Proof of part (b) is similar. Q.E.D.

<u>Theorem 12:</u>

$$E_1 \overline{\vee} E_2 = \bigcup_{e_j \in E_1} \bigcap_{e_i \in \overline{E}_2} (\{e_j\} \overline{\vee} \{\overline{e_i}\})$$

where the order of the union and intersection is irrelevant.

<u>Proof:</u>

$$E_1 = \bigcup_{e_j \in E_1} \{e_j\} \text{ and } E_2 = \bigcup_{e_i \in E_2} \{e_2\} =$$

$$= \bigcap_{e_i \in \overline{E}_2} \{\overline{e_2}\}. \text{ Thus:}$$

$$E_1 \overline{\vee} E_2 = \bigcup_{e_j \in E_1} \{e_j\} \overline{\vee} \bigcap_{e_i \in \overline{E}_2} \{\overline{e_i}\}$$

theore

$$\sqrt[\vee]{X_1^{a\,b\,c}\, X_2^{\alpha}} = \{ X_1^{a\,b},\ \overline{X_2^{c\,\alpha}} \}$$

Apply now the distribution rules of the theorem 11 (in any order) to derive the desired result. Q.E.D.

The theorem 12 gives a theoretical rule to compute the set L_λ of all maximal intervals in f under λ. Namely:

$$L_\lambda = \sqrt[\cup]{F^{1\lambda} \cup \overline{F^{0\lambda}}} \quad (33)$$

$$L_\lambda^{\cup} = F^{1\lambda} \cup \overline{F^{0\lambda}} = \bigcup_{e_j \in F^{1\lambda}} \bigcap_{e_i \in F^{0\lambda}} (e_j \cup \overline{(e_i)}) \quad (34)$$

Find the irredundant expression of $L_\lambda^{\cup}$ as a union of intervals. Set L_λ is the set of intervals in this expression. A more detailed description of the above procedure - in the case when $F^{1\lambda}$ consists of the only one element - is given in the next section.

5.2 Definition of a star $G(e|\lambda)$ and an algorithm G for its generation.

A fundamental concept in our approach to the synthesis of interval covers is of an interval star $G(e|\lambda)$.

Definition 11. The interval star $G(e|\lambda)$ under λ of an event $e \in F^{1\lambda}$ is a set of all maximal intervals under λ which cover the event e, i.e.

$$G(e|\lambda) = \{L_k^\lambda \mid e \in L_k^\lambda\} \quad (35)$$

We denote $G(e|1)$ by $G(e)$. $G^{\cup}(e|\lambda)$ is - according to (25) - the union of intervals in $G(e|\lambda)$. An interval star can be expressed as:

$$G(e|\lambda) = \sqrt[\cup]{\{e\} \cup \overline{F^{0\lambda}}} \quad (36)$$

Theorem 13:
$$G^{U}(e|\lambda) = \bigcap_{e_i \in F^{0\lambda}} (\{e\} \vee \{\overline{e_i}\})$$

Proof: $G^{U}(e|\lambda) = \{e\} \vee \overline{F^{0\lambda}} = \{e\} \vee \bigcap_{e_i \in F^{0\lambda}} \{\overline{e_i}\}$

Applying now theorem 9 part (a) we complete the proof. Q.E.D

The following algorithm for the generation of a star $G(e|\lambda)$ follows from theorem 13 (algorithm G).

Given $e \in F^{1\lambda}$ and $F^{0\lambda} = \{e_i\}_{i=1}^{z}$.

Algorithm G:

1. Determine for $i = 1,2,\ldots,z$ $\{\overline{e_i}\}$ and then form $D_i = \{e\} \vee \{\overline{e_i}\}$
2. Set up the function $\bigcap_{i=1}^{z} D_i$ and find its irredundant expression by multiplying each term D_i by all the others and applying the absorption laws.
3. $G(e|\lambda)$ is the set of terms of the expression thus obtained.

5.3 Synthesis of a cover $M^{q}(f|\lambda)$ by algorithm A^{q}

The problem of synthesis of a minimal cover $M(f|\lambda)$ is a particular case of the general covering problem described in [2]. As indicated in this paper even in case of a relatively simple covering problem the number of operations required for its exact solution may not be feasible even with the fastest computers. Consequently, the most desirable are good methods for an approximate solution which allow to drastically reduce the number of operations but also give some measure of distance to the minimum.

The previously mentioned method of disjoined stars, when realized by locally optimal decisions, generates--with a relatively small number of operations and memory requirements--a so-called quasi-minimal cover, which is either minimal or approximately minimal. Furthermore, when we cannot state that obtained solution is minimal, it provides an estimate of the maximal possible distance between the obtained solution and the minimal one.

The fundamental theorem from which this method stems (expressed in our terms) is following.

Let us assume that we are given (by realizing some algorithm) a family of stars $G^{r} = \{G_i(e|\lambda)\}$, $e \in F^{1\lambda}$, such that any two stars chosen from it are disjoint sets (we say G^{r} is a family of disjoint stars).

Theorem 14: The number of intervals in the minimal cover $M(f|\lambda)$ satisfies the relation:

$$c(M(f|\lambda)) \geq c(G^r)$$

The theorem implies that if we have a cover $D(f|\lambda)$ and know the number $\rho = c(G^r)$, then the difference $\Delta = (D(f|\lambda)) - \rho$ can be viewed as an estimate of the difference between the number of elements in this cover, and in a minimal one. If we can next find another family of disjoint stars with a greater number of elements, say ρ_1, then we can improve our estimate, namely:

$$\Delta_1 = (D(f|\lambda)) - \rho_1 < \Delta$$

If on the other hand we can find another cover with a smaller number of intervals, then obviously our estimate will also improve, and it may turn out then Δ will become 0. This will mean that we found the minimal solution.

A possible algorithm for accomplishing the above ideas with a view toward a solution of the generally stated covering problem, was described in paper [2] (algorithm A^q). In the formulation given there it was assumed that a cover consists of some sets (not concretely specified), called complexes. In our case the complexes are specified as intervals (interval complexes). Thus in order to apply the algorithm A^q for our purpose we only need to make use of the algorithm for generating a star $G(f|\lambda)$, described in section [5.2].

The flow diagram of algorithm A^q in the form adapted to our notation is shown in fig. 2.

The sign := denotes assignment statement (as in Algol 60). F^p is an auxiliary variable. $O(F^p; e_1)$ denotes the operation of choosing the event with the smallest number from the set specified by the current value of F^p and assigning the notation e_1 to that event. L^q denotes an interval in $G(e_1|\lambda)$, called a quasi-extremal, which covers the maximum number of events in the set constituting the current value of variable $F^{1\lambda}$. It can be noticed that this is a locally optimal decision about the choice of an interval from the given star (generally not unique). The last value of M^q constitutes our solution, the quasi-minimal cover $M^q(f|\lambda)$. Value Δ is an estimate of the maximal possible difference between the cover $M^q(f|\lambda)$ and a minimal cover $M(f|\lambda)$, expressed in number intervals, i.e.

$$c(M^q(f|\lambda)) - c(M(f|\lambda)) \leq \Delta. \tag{27}$$

If after the first execution of the algorithm, Δ is considered to be too large, the better estimate (and/or solution) may be obtained realizing next iterations, e.g. in the way described in [3].

To synthesize a quasi-minimal free cover $M^q(f|\lambda,p)$ using this algorithm, we substitute in the flow diagram (fig. 2) the set $F^{1\lambda}$ by $F^{1\lambda p}$, which means that we have to cover all elements from $F^{1\lambda} \setminus F^{\phi i(\lambda)}$ and some elements (at least one) from the set $F^{1\lambda} \cap F^{\phi i(\lambda)}$.

A geometrical interpretation of interval covers using a generalized logical diagram and examples of interval covers synthesis are given in [5]. An extension of the algorithm for the synthesis of quasi-minimal ordered covers and application of the concepts described in the paper to pattern recognition are given in [6].

6. REMARKS ON APPLICATIONS

A cover of a mapping f consisting of multidimensional intervals can be interpreted as a set of 'filters' for recognizing events from a signal class (represented by $F^{1\lambda}$) for pattern recognition and picture processing purposes, e.g. for discriminating regions of different textures, striping of background, local feature extraction, border detection, etc.

In case of an ordered cover the individual filters correspond to the consecutive points on the optimum receiver-operating-characteristic (ROC), defined as in statistical decision theory. The above ROC curve can be obtained by optimal ordering events from the two classes to be distinguished with regard to the likelihood ratio of their frequency occurence [6].

7. REFERENCES

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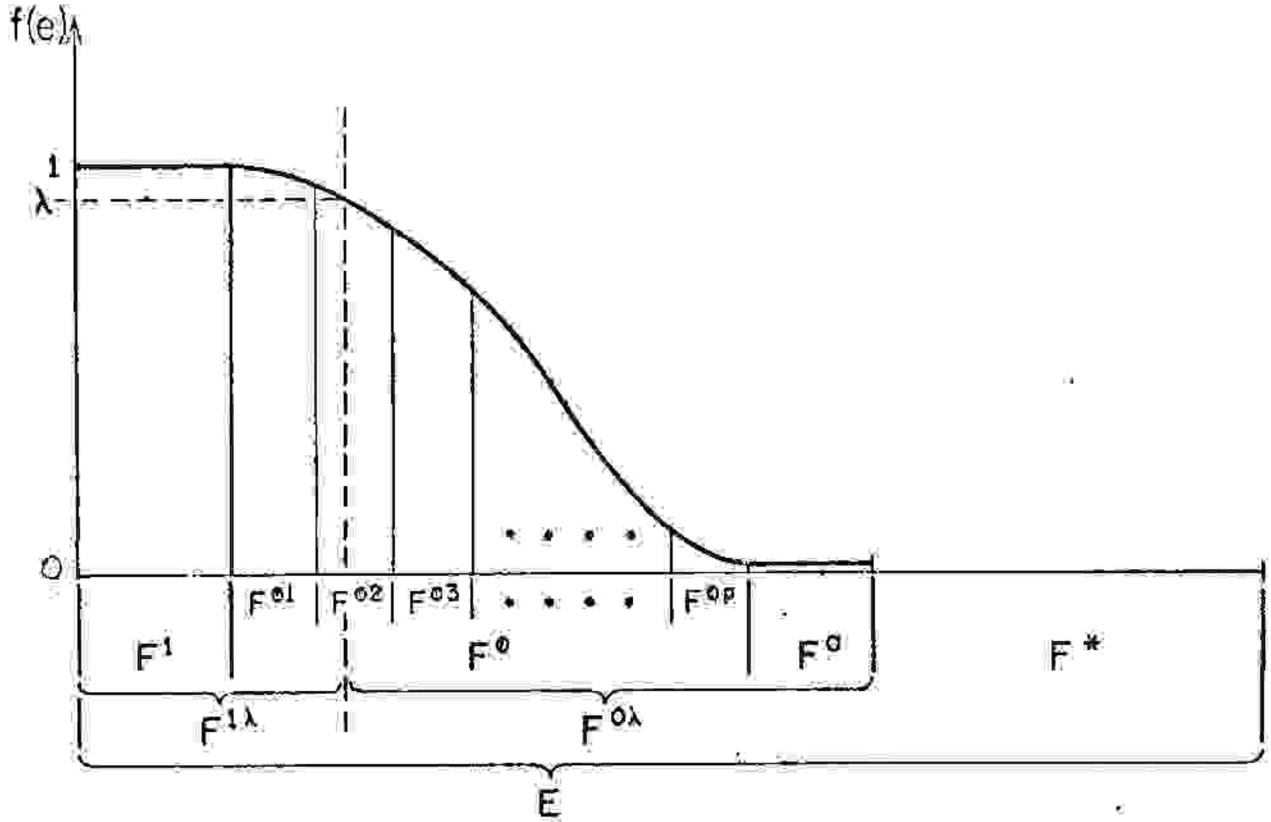

Fig. 1

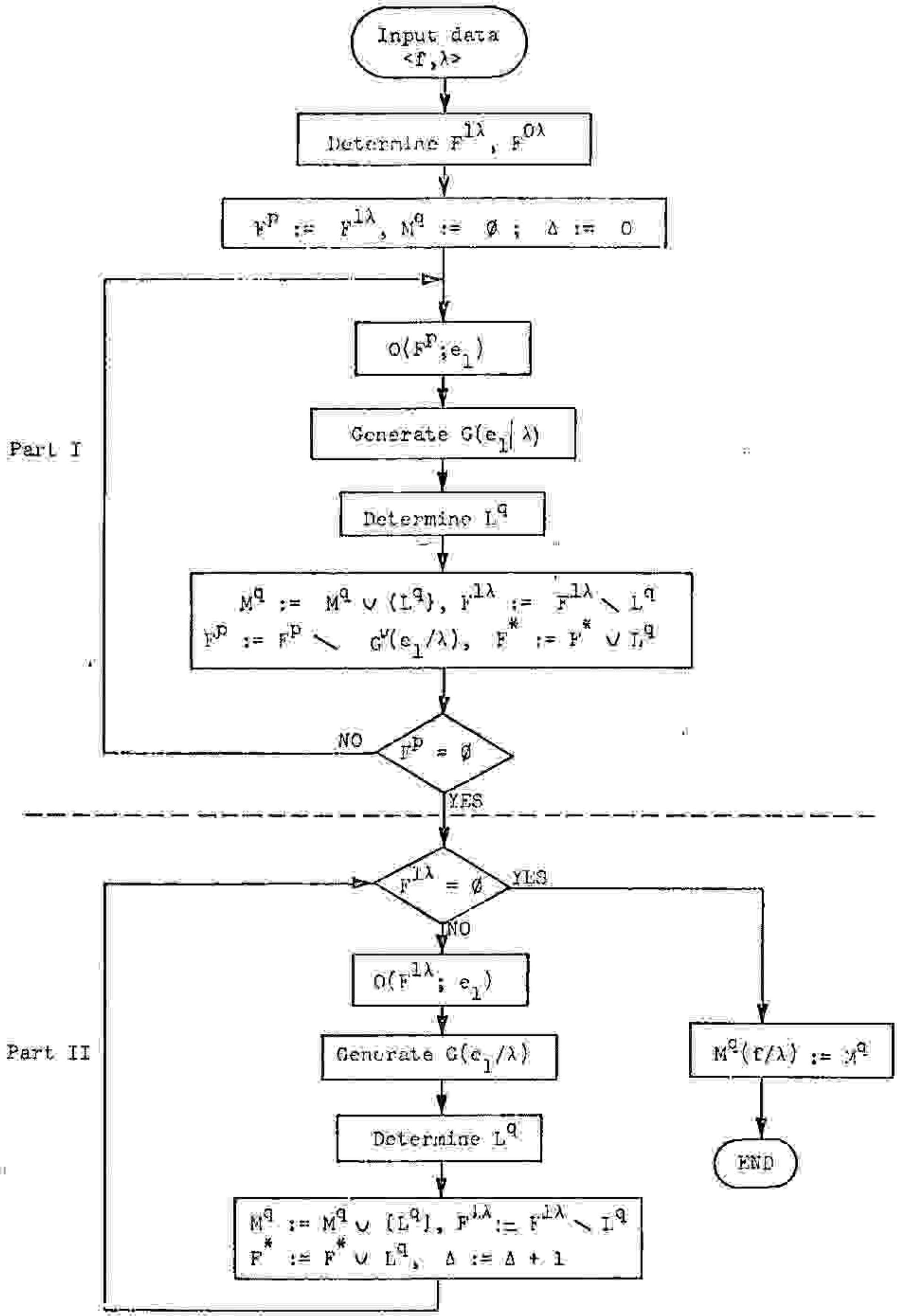
Input data
<f,λ>
Determine $F^{1\lambda}$, $F^{0\lambda}$
$F^p := F^{1\lambda}$, $M^q := \emptyset$; $\Delta := 0$
$O(F^p; e_1)$
Generate $G(e_1/\lambda)$
Determine L^q
$M^q := M^q \cup \{L^q\}$, $F^{1\lambda} := F^{1\lambda} \setminus L^q$
$F^p := F^p \setminus G^{\nu}(e_1/\lambda)$, $F^* := F^* \cup L^q$
NO
$F^p = \emptyset$
YES
Part I
$F^{1\lambda} = \emptyset$
YES
NO
$O(F^{1\lambda}; e_1)$
Generate $G(e_1/\lambda)$
Determine L^q
$M^q := M^q \cup \{L^q\}$, $F^{1\lambda} := F^{1\lambda} \setminus L^q$
$F^* := F^* \cup L^q$, $\Delta := \Delta + 1$
$M^q(f/\lambda) := M^q$
END
Part II

Fig. 2