Subjective foundation of possibility theory: Anscombe-Aumann approach

Phan H. Giang

George Mason University, MS 1J3
4400 University Dr., Fairfax, VA 22030, USA
pgiang@gmu.edu

Abstract

This paper presents a new characterization of subjective possibility theory. Anscombe and Aumann [1] pioneered a multiple-variable approach to characterize the subjective probability on the basis of its relationship with objective probability. We generalize the Anscombe-Aumann approach to define subjective possibility theory. To characterize the uncertainty of the variable of interest (H) we consider its relationship with a variable of probability and a variable of ignorance. Our axiomatization assumes that the preference over the acts contingent on the outcomes of the variables is a weak order and satisfies a monotonicity property; the preference over probabilistic lotteries is represented by linear utility; and the preference over acts under ignorance by a recently developed τ-anchor utility theory. We show that if H and the variable of ignorance are completely reversible in Anscombe-Aumann's sense and H and the variable of probability are partially reversible then the uncertainty of H must obey the laws of possibility theory. We outline a framework to manage heterogeneous uncertainty that includes probability, possibility and ignorance in decision making.

Keywords: Possibility theory, Anscombe-Aumann approach, ignorance, τ-anchor utility, order between variables, heterogeneous uncertainty.

1. Introduction

Possibility theory is an uncertainty theory designed to capture important and intuitive notions that elude probability theory, for example, partial ignorance in knowledge and similarity measure between entities. Possibility theory has been successfully used in many practical applications and applied to address theoretical issues such as knowledge representation, reasoning under uncertainty, control and planning. However, the mainstream AI community still employs, almost exclusively, probability theory to express uncertainty. Thirteen years ago, Artificial Intelligence Journal dedicated a special issue "Fuzzy set and possibility theory-based methods in artificial intelligence" (v.148, 2003) to bridge the gap between mainstream AI community and the fuzzy set-possibility theory community. Despite such efforts, it is safe to say that not only the gap still exists but it becomes even wider. One of the reasons behind this unsatisfactory situation, in our judgment, is the lack of consensus on the issues pertinent to the foundation of possibility theory and especially, its linkage to the choice behavior of individuals.

Mathematically, a possibility measure Π is a function $\Pi : \Omega \to [0,1]$ on state space Ω such that $\Pi(\Omega) = 1$ and

$$\forall A, B \subseteq \Omega, \Pi(A \cup B) = \max(\Pi(A), \Pi(B)). \tag{1}$$

In literature, it is argued that the meaning of Π can be derived from Zadeh's fuzzy set theory [20], statistical evidence [15].

The proponents of possibility theory have done a good job explaining how possibilistic uncertainty is different from probabilistic risk. At the same time, not enough attention was paid to the questions such as how possibility theory is founded on individual choice behavior and how possibility is used in conjunction with probability. This gap of understanding has stymied wider adoption of possibility theory in practice.

This paper will address some of outstanding foundational issues of possibility theory. We provide a new decision theoretic foundation for possibility theory using Anscombe-Aumann's approach. Unlike previous works that examined possibilistic uncertainty in isolation, we consider it in conjunction with other types of uncertainty, namely, probability and ignorance.[1]

This approach offers two-fold advantage. It provides an interpretation of possibility in terms of well-understood uncertainty and a framework in which different types of uncertainty can be expressed, reasoned about and used for decision making.

[1]Following a tradition in computer science where data types are distinguished by their operations, by uncertainty *type* we mean the set of laws that regulate the uncertainty not the epistemic nature of the uncertainty. In this sense, objective probability and subjective probability are of the same type (i.e., risk) while possibility and probability are different types because they obey different sets of rules.

In a classic paper "A definition of subjective probability" [1], Anscombe and Aumann (AA) derived subjective probability representation from the notion of objective probability and postulates of rational choice behavior. This result, together with Savage's landmark "The Foundations of Statistics" [13], provides a solid foundation for subjective (personal) probability. In this paper, following AA approach, we aim to define subjective possibility theory.

The paper is structured as follows. In Section 2, we briefly review AA approach to subjective probability. In Section 3, we review decision making under complete ignorance. Section 4 presents the main result. Section 5 presents an algorithm to compute the certainty equivalent of acts under subjective possibility. Section 6 has a running numerical example. Section 7 presents the discussion of related literature. The last section has concluding remarks.

2. Anscombe-Aumann approach to subjective probability

AA original framework considers two types of uncertain variables. On the one hand, the variable of objective probability, denoted by R, represents the outcomes generated by mechanical devices or natural processes, such as roulette spins. A characteristic for this type of variable is the ability to repeat the experiments in identical conditions, hence, the ability to estimate the chances of events based on frequency. On the other hand, the variable of subjective uncertainty, denoted by H, represents the outcomes of an uncertain event, such as a horse race, which is in some sense unique because it is impossible to replicate all the conditions that influence the outcomes. For a particular horse race, due to the absence of data of identical races, an individual must form her subjective belief based on the available information and her judgment. AA showed that if her judgment satisfies some normative rules then her subjective belief must obey the laws of probability.

Assume that $\mathcal{O}$ is the set of prizes. A roulette lottery is a tuple $(p_1 : x_1, p_2 : x_2, \ldots, p_n : x_n)$ where $x_i \in \mathcal{O}$ and p_i is the (roulette) chance of getting x_i. Suppose that a horse race can realize in one of m outcomes $s_1, s_2, \ldots, s_m$, a horse lottery, denoted by $[s_1 \mapsto y_1, s_2 \mapsto y_2, \ldots, s_m \mapsto y_m]$, is a contract that delivers roulette lottery y_j if s_j realizes. The basic objects in AA theory are the roulette lotteries whose prizes are the horse lotteries. Thus, such compound lotteries have 3 stages where the uncertainty in two stages is described by objective probability (the roulette spins) while the uncertainty in the other stage is subjective belief (the horse race). AA approach characterizes the subjective uncertainty based on the objective risk and several rational assumptions imposed on preference between lotteries.

The first assumption states that the preference among lotteries is complete and transitive. The second assumption states that the preference among roulette lotteries is described by expected utility. The third assumption, called "Monotonicity in prizes", requires that substitution of a roulette lottery in horse lottery by a preferred lottery yields a preferred horse lottery. Formally, if $y'_j \succeq y_j$ then

$$[s_1 \mapsto y_1, \ldots, s_j \mapsto y'_j, \ldots, s_m \mapsto y_m] \succeq [s_1 \mapsto y_1, \ldots, s_j \mapsto y_j, \ldots, s_m \mapsto y_m].$$

Finally, the fourth assumption "Reversal of order in compound lotteries" requires that reversing the order in which uncertainty is resolved does not change the "value" of the compound lottery. Formally,

$$\begin{aligned} &(p_1 : [s_1 \mapsto y_{11}, \ldots, s_m \mapsto y_{1m}], \ldots, p_n : [s_1 \mapsto y_{n1}, \ldots, s_m \mapsto y_{nm}]) \\ \sim &[s_1 \mapsto (p_1 : y_{11}, \ldots, p_n : y_{n1}), \ldots, s_m \mapsto (p_1 : y_{1m}, \ldots, p_n : y_{nm})]. \end{aligned}$$

The above pair of indifferent lotteries describes two different situations. The former describes a situation in which roulette is spun first and depending on its outcome the individual-owner of the lottery is given a horse lottery. When the horse race is completed, depending on its outcome the individual is rewarded with a roulette lottery y_{ij}. On the other hand, the latter describes a situation in which the order between roulette spin and horse race is reversed. The horse race is run first, and depending on its outcome, the individual is given a roulette lottery that delivers a reward y_{ij} depending on the outcome of a roulette spin. We have the following theorem.

Theorem 1 (Anscombe-Aumann 1963). *Given the assumptions, there is a unique set of non-negative numbers $q_1, q_2, \ldots, q_m$ summing up to 1 such that for any pair of horse lotteries*

$$[s_1 \mapsto y_1, \ldots, s_j \mapsto y_j, \ldots, s_m \mapsto y_m] \succeq [s_1 \mapsto y'_1, \ldots, s_j \mapsto y'_j, \ldots, s_m \mapsto y'_m] \textit{ iff}$$
$$\sum_{j=1}^{m} q_j u(y_j) \geq \sum_{j=1}^{m} q_j u(y'_j)$$

where u is the linear utility function describes the preference among roulette lotteries.

It is hard to avoid an impression that with this theorem, Anscombe-Aumann managed to pull a rabbit out of a hat. There is no informational dependency between the roulette spin and the horse race in the sense that knowing the outcome of one has no influence on the belief about the outcome of the other. So, the fact that uncertainty the individual has about a horse race obeys the laws of probability is derived exclusively from the reasonable assumptions imposed on her preference among compound lotteries, in particular, the "Monotonicity in prizes" and "Reversal of order". A contra-positive consequence of AA theorem is that, if the uncertainty of a variable does not satisfy the probability laws then one of the assumptions must be violated. In section 7, we discuss deviations from the AA model.

3. A utility theory under ignorance

This section offers a review of a utility theory for decision making under ignorance. The material is drawn from [8]. Traditionally, uncertainty in most applications is described in the language of probability, so it is necessary to discuss what is ignorance and why we have to consider it.

Ignorance is a singular state of knowledge characterized by total lack of knowledge and/or reliable information about the phenomenon of interest. Under this extreme state of uncertainty, it is impossible to compare the likelihoods between two events except tautology and null events. In practice, ignorance can arise when information is scarce, unreliable and contradictory. Ignorance exists in competitive games where information about the opponent is intentionally misleading. Two key conditions that distinguish ignorance from uncertainty is (1) the absence of the knowledge about the factors that influence the issue [12] and/or (2) inability to determine the space of alternatives [3]. Being an extreme form of uncertainty, ignorance eludes probability theory. For example, ignorance cannot be represented by a uniform distribution when the sample space is ill-determined. It can not be adequately represented by a personal probability measure when one lacks the understanding of factors that drive the outcomes. In fact, the quest to capture ignorance and to reason about it is a major motivation to develop non-probabilistic representations of uncertainty. In reality, one normally has to deal with epistemic situations where "pockets of ignorance" coexist with the knowledge. An analysis of such situations would naturally lead to multiple-variable representation of the domain of interest that separates the pockets of ignorance from knowledge.

Let's borrow an example given in [8]. An investor is considering at the end of 2015 an investment instrument that matures on 1 January 2018. The return on the investment depends on two uncertain variables. The first source of uncertainty is the prospect of a political settlement in country A (e.g., Afghanistan) and the second source of uncertainty is the prospect of 2017 coffee crop in country B (e.g. Brazil). The 2017 coffee harvest, denoted by B, can be either bumper (b) or normal (n) or poor (p). On the one hand, from extensive historical data, the probability distribution of Brazilian coffee crop in 2017 is estimated to be $(0.46, 0.2, 0.34)$ where the numbers are the chances of having bumper, normal and poor crop respectively. On the other hand, the political settlement variable, denoted by A, is modeled with two possible values: peaceful settlement among fighting factions (s) or lack thereof (s̃). The experts whose advice the investor seeks on the political settlement question, offer contradictory opinions and cannot come to any agreement. This underlies the fact that nobody knows the true driving forces behind a political settlement in that region of the world.[2] The returns on the investment are given in the following table.

A/B	b	n	p
s	-3.0%	2.0%	8.0%
s̃	5.0%	1.5%	-4.0%

Table 1: The returns on investment.

For example, if there will be a bumper coffee crop and a political settlement then the the investment has negative return of -3%. The question for the investor is whether she should invest in the financial instrument or keep her money in the bank that pays interest of 1.2%. In Section 6 we extend this example to include a subjective possibility variable and show the steps needed to answer investor's question.■

In [8], a utility theory is developed that combines Hurwicz-Arrow's theory of decision under ignorance [2] with Anscombe-Aumann's ideas of reversibility and monotonicity that had been used to characterize subjective probability. The main result is a representation theorem for preference under

[2]If you find the portrait of Afghanistan situation a bit too unrealistic, an article "America's Shocking Ignorance of Afghanistan" in *The National Interest Magazine* (06/05/2015) could change your opinion. http://nationalinterest.org/feature/americas-shocking-ignorance-afghanistan-13049.

ignorance by a special one-parameter function – the τ-anchor utility function.

Let's consider a collection of variables of ignorance $\{I_1, I_2, \ldots\}$ whose domains are sets Ω_{I_i} which are assumed to be finite subsets of the set of natural numbers $\mathbb{N}$. A *decision* or *act* defined on variable I_i is a mapping $f : \Omega_{I_i} \to \mathcal{O}$ where $\mathcal{O}$ is the set of *prizes* (rewards) which is assumed to be the real unit interval. The set of finite subsets of elements in $\mathcal{O}$ is denoted by $\mathcal{F}(\mathcal{O})$. The domain of f is denoted by $\Omega(f)$. The set of such acts is denoted by $\mathcal{D}_I$. A preference relation $\succeq_I$ on $\mathcal{D}_I$ is assumed to be a weak order (reflexive, complete and transitive). The symmetric (indifference) part, $(\sim_I)$ and the asymmetric (strict preference) part $(\succ_I)$ of $\succeq_I$ are defined as usual: $f \sim_I g \Leftrightarrow f \succeq_I g \wedge g \succeq_I f$ and $f \succ_I g \Leftrightarrow f \succeq_I g \wedge g \not\succeq_I g$.

In a short paper reprinted in 1977, Hurwicz and Arrow [2] described a theory of decision under ignorance that was developed in early 1950s. Preference $\succeq_I$ must satisfy four axioms named after letters A to D listed below. Before listing the axioms, let us recall two definitions. Two acts f_1 and f_2 are *isomorphic* if there is a one-to-one mapping h from the domain of f_1 to the domain of f_2 such that $\forall s \in \Omega(f_1), f_1(s) = f_2(h(s))$. h is also called the *relabeling* operation. Act f_2 is said to be *derived from* act f_1 by deleting duplicate states if (1) $\Omega(f_2) \subset \Omega(f_1)$ and f_1 and f_2 are coincide on $\Omega(f_2)$ and (2) for each $w \in \Omega(f_1) - \Omega(f_2)$, there exists $w' \in \Omega(f_2)$ such that $f_1(w) = f_1(w')$.

(A) (Weak order). $\succeq_I$ is a weak order.

(B) (Invariance under relabeling axiom (symmetry)). If acts are isomorphic then they are indifferent.

(C) (Invariance under deletion of duplicate states). If f_2 is derived from f_1 by deleting duplicates then f_1 and f_2 are indifferent.

(D) (Weak dominance axiom). If f_1, f_2 are acts on the same domain Ω_f and $\forall w \in \Omega_f, f_1(w) \geq f_2(w)$ then $f_1 \succeq_I f_2$.

Axioms A and D belong to the group of standard assumptions of rational behavior. Axioms B and C capture the essence of the notion of ignorance. For example, no probability representation would satisfy both B and C. The axioms also hint about epistemic difficulty that leads to ignorance, for example, inability of the individual to determine the "right" state space. Axioms B and C make the whole notion of state space irrelevant for preferential evaluation of acts. Thus, a generic domain of natural numbers can be used and furthermore, acts can be identified with the sets of the prizes (members of $\mathcal{F}(\mathcal{O})$). The main result is a representation theorem.

Theorem 2 (Hurwicz-Arrow). *The necessary and sufficient condition for preference $\succeq_I$ on the set of acts $\mathcal{D}$ satisfies properties A through D is that*

$$f \succeq_I g \text{ whenever } \underline{f} \geq \underline{g} \text{ and } \overline{f} \geq \overline{g} \tag{2}$$

where $\underline{f}$ and $\overline{f}$ denote the minimal and maximal value of function f.

HA theorem says that the comparison between two sets of prizes reduces to comparison of their extrema. If both extrema of a set are greater than or equal to their counterparts in the other set then the former is preferred to the latter. The intermediate members of the set do not matter. The theorem does not specify the preference between two acts when their prize ranges are nested one inside the other. A value $x \in \mathcal{O}$ is called the *certainty equivalent* (CE) of act A if $A \sim_I x$ (notation $x = \mathcal{CE}(A)$).

The multiple-variable approach is also useful in understanding the decision under ignorance. Let's consider two ignorant variables I_1 and I_2 that are *independent*.[3] It is arguable that if an individual is ignorant about both I_1 and I_2 and the variables are independent then the person is ignorant about the joint variable I_{12} obtained by collapsing I_1 and I_2 i.e., $\Omega_{I_{12}} = \Omega_{I_1} \times \Omega_{I_2}$.

Thus, AA argument for adoption of Monotonicity and Reversal of order axioms can be translated to apply for the preference under ignorance. To fix the idea let's assume that the domains of I_1 and I_2 have only two elements $\Omega_{I_1} = \{s_{11}, s_{12}\}$ and $\Omega_{I_2} = \{s_{21}, s_{22}\}$. A lottery on single variable is a two-element set $\{x, y\}$ with $x, y \in \mathcal{O}$. A compound lottery is a collection of two sets $\{q_1, q_2\}$ where $q_1 = \{x_1, y_1\}$ and $q_2 = \{x_2, y_2\}$. We can now state the assumptions of Monotonicity under ignorance (MI) and Reversal of ignorant variables (RI).

(MI) $\{q_1, q_2\} \succeq \{q'_1, q'_2\}$ if $q_i \succeq q'_i$ for $i = 1, 2$.

(RI) $\{\{x_1, y_1\}, \{x_2, y_2\}\} \sim \{\{x_1, x_2\}, \{y_1, y_2\}\}$.

In [8] it has been showed that if on top of four HA axioms, (MI) and (RI) axioms are adopted then it is possible to characterize a preference under ignorance by a one-parameter utility function of special form.

[3] In this paper we discuss uncertainty that is not necessarily probabilistic in nature. The independence between variables is understood in the sense of "freedom from control or influence of one another". That is, knowing the outcome of one variable does not give any information or change the belief about the other. When both variables are probabilistic, this notion of independence reduces to the traditional stochastic independence.

Theorem 3 (Giang 2015). *Let $\succeq$ be a HA preference relation on the acts defined on two ignorant variables I_1 and I_2 that are also independent. Under some technicality conditions, $\succeq$ satisfies Monotonicity under ignorance (MI) and Reversal of ignorant variables (RI) assumptions iff there exists a value $\tau \in [0,1]$ such that the certainty equivalence operator $\mathcal{CE}$ of $\succeq$ has the form: for $A_i \in \mathcal{F}(\mathcal{O})$, $1 \leq i \leq m$*

$$\mathcal{CE}(A_i) = \begin{cases} \max(A_i) & \text{if } \max(A_i) < \tau \\ \tau & \text{if } \min(A_i) \leq \tau \leq \max(A_i) \\ \min(A_i) & \text{if } \min(A_i) > \tau \end{cases}, \quad (3)$$

$$\mathcal{CE}(\cup_{i=1}^{m} A_i) = \mathcal{CE}(\{\mathcal{CE}(A_i) | 1 \leq i \leq m\}). \quad (4)$$

τ is called the *characteristic value* under ignorance and it completely describes the behavior under ignorance of an individual. Note that (3) describes the certainty equivalence for a single-stage lottery while (4) describes CE for a compound two-stage lottery under ignorance. The CE of a set of prizes under ignorance is a point in the interval spanning the minimum and the maximum of the prize set that minimizes the distance to the characteristic value. Imagine an ideal rubber cord with one end fixed to τ and the other end is allowed to move freely within the interval covering the set of prizes. The CE is the point where equilibrium is attained. The function (3) is called the τ-*anchor* utility function. Note that functions $\max, \min$ are special cases when $\tau = 1$ and $\tau = 0$ respectively. Also the functional form (3) is also known as the null-norm aggregation function [11].[4]

For example, from the equality $\tau = \mathcal{CE}(\{0,1\})$ it follows that τ is the value that the decision maker would give in exchange for the entire set of prizes $[0,1]$. This is a situation of total ignorance. Not only the decision maker is ignorant about the likelihood of variable realization but also no prize in the set of possible prizes is excluded. The values of τ are used to compare the extent of the *tolerance for ignorance* between different individuals. For two individuals A and B with characteristic values τ_A and τ_B, we say that A is *more tolerant for ignorance than* B if $\tau_A \geq \tau_B$.

We say that two uncertain variables are *collapsible* if every two-stage act contingent on their outcomes can be reduced to an equivalent one-stage act on the variable of the joint domain (Cartesian product of their domains), moreover, the uncertainty measure on the joint domain is constructed from

[4] I thank a reviewer for pointing out this link.

the uncertainty measures of original variables. In the example of investment options that depend on both the Brazilian coffee crop and Afghanistan peace settlement, the collapsibility would mean that there is an uncertainty measure on the joint variable "CoffeePeace" that can be constructed from the probability distribution on coffee crop outcomes and the ignorance about peace settlement outcomes. We have seen that like a pair of independent probabilistic variables, two independent ignorant variables are collapsible and reversible. However, it has been shown in [8] that an ignorant variable and a probabilistic variable are not reversible even if they are independent.

4. Probability, complete ignorance and possibility

In this section, AA framework is extended to consider situation that involves three types of uncertainty. In the subsection 4.1 we discuss the assumptions. The main result about the characterization of possibilistic uncertainty is developed in subsection 4.2.

4.1. Framework and assumptions

To be specific, we name our variables as R (for roulette spin), I (for ignorance) and H (for horse race). The uncertainty structures of R and I are known. R is probabilistic and I is ignorance (as discussed in the previous section). The uncertainty structure H is subjective. The aim is to specify the laws that subjective uncertainty of H must obey. We'll show that rational postulates that govern the relationships between two pairs of variables (H, I) and (H, R) imply that uncertainty of the horse race satisfies the axioms of possibility theory.

The domains of variables are denoted by Ω_R, Ω_I and Ω_H. We assume they are mutually independent. An order of variables is a (chronological) sequence of variables. For example, $R \lhd I \lhd H$ is an order where R precedes I which precedes H.

We consider acts whose rewards are contingent on the outcomes of the three variables. We distinguish two concepts of acts. *Functional acts* are mappings of the form $f : \Omega_R \times \Omega_I \times \Omega_H \to \mathcal{O}$. A *sequential act* is a functional act under a specific order of variables. The graphic representation of a sequential act is a tree with its branches labeled by variable values and its terminal nodes labeled with prizes. Alternatively, a functional act corresponds to a *set* of sequential acts associated with the permutations of variable order. Keep in mind that "act" can be either functional or sequential and the ambiguity is resolved by context.

The distinction between a functional act and a sequential act is not material for decision under risk and, for that matter, decision under ignorance because the probabilistic (ignorance) variables are fully reversible so they are collapsible into one variable. However, in general, because the reversibility does not holds universally among variables of different types, a sequential act is materially different from its functional counterpart.

We introduce the concept of *effective dimension* of an act. Suppose that f is an act on three variables R, I and H. We say that H is a *spurious* variable if $\forall x, x' \in \Omega_H$, $\forall r \in \Omega_R$, $\forall a \in \Omega_I$, $f(r, a, x) = f(r, a, x')$. If H is spurious then f is called a 2D act with effective dimensions R and I (notation RI-act). When the prizes effectively depend on a single variable the act is 1D . We have an inclusion chain: the set of 3D acts includes the set of 2D (RI-, RH- and IH-) acts which in turn includes the set of 1D (R-, I- and H-) acts which finally includes the set of prizes that have zero dimension. We reserve term *lottery* for a probabilistic act (R-act).

Denote by $\mathcal{D}$ the set of all sequential acts. We consider a preference relation $\succeq$ on $\mathcal{D}$ that is a weak order (complete, transitive and reflexive). The relation $\succeq$ is called the *mother* preference relation and can be projected on the sets of 2D or 1D acts. The daughter preference relations are denoted by the index of dimensions. For example, the projection on the set of IR-acts is denoted by $\succeq_{IR}$. Formally, if f, g are IR-acts then $f \succeq g \Leftrightarrow f \succeq_{IR} g$. Because all the daughter relations are from the same mother preference relation, they are consistent. The preference symbols can be used flexibly depending on the context when more than one symbols are applicable. For example, for prizes $x, y \in \mathcal{O}$ we can write $x \geq y$ or $x \succeq_R y$ or $x \succeq y$.

Assumption 1 (Representation of 1D acts (OD)).
(i) The derived preference on lotteries, $\succeq_R$, is represented by a strictly monotone linear utility;
(ii) The derived preference on ignorance acts, $\succeq_I$, is represented by the τ-anchor utility.

We can assume that the strictly monotone utility function u is normalized such that $u(0) = 0$ and $u(1) = 1$. Also, ignorance acts are evaluated by τ-anchor utility. The representation of derived preference on H-acts, $\succeq_H$, is to be characterized. Given this assumption, by theorem 3, I-acts can also be identified with *bags* of prizes denoted by a list enclosed in a pair of *curly brackets* e.g., $\{x_1, x_2, \ldots, x_m\}$. The set of such bags is $\mathcal{F}(\mathcal{O})$. R-acts or *lotteries* are denoted by lists of probability-prize pairs enclosed in a couple of *round brackets*, for example, $(p{:}x, 1-p{:}y)$. The set of lotteries is denoted by $\mathcal{L}(\mathcal{O})$.

For H-acts we use three notational forms to help readability. Suppose $\Omega_H = \{w_1, w_2, \ldots, w_n\}$. H-act can be described by a list of n elements enclosed in *square brackets* $[x_1, x_2, \ldots, x_n]$ where x_i is the prize in case of w_i. This is the *matrix* notation. When the states associated with prizes are relevant to discussion, we use the *rule* notation $[A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_k \mapsto x_k]$ where $\{A_i\}_{i=1}^k$ is a partition of Ω_H. When the partition is binary $\{A, \bar{A}\}$, the *compact* notation A_y^x is used instead of $[A \mapsto x, \bar{A} \mapsto y]$.

The notational convention is extended for 2D or 3D sequential acts accordingly. For example, $\{(p : x, 1-p : y), (q : x, 1-q : y)\}$ denotes a bag of lotteries and $(p : \{x, y\}, 1-p : \{u, v\})$ is a lottery on bags of prizes. The correct reading of expressions uses the following precedence rule: brackets $\Rightarrow$ arithmetic operations $\Rightarrow$ ":" (used to separate the probability from the prize components) $\Rightarrow$ "," (used to separate elements of a list).

The next assumption says that $\succeq$ must satisfy Monotonicity.

Assumption 2 (Monotonicity (M)). *Suppose $z_i, z_i' \in \mathcal{D}$ for $1 \le i \le n$ and the sequential acts below are in $\mathcal{D}$, if $z_i \succeq z_i'$ then*

$$[z_1, z_2, \ldots, z_n] \succeq [z_1', z_2', \ldots, z_n'] \tag{5}$$

$$\{z_1, z_2, \ldots, z_n\} \succeq \{z_1', z_2', \ldots, z_n'\} \tag{6}$$

$$(p_1{:}z_1, p_2{:}z_2, \ldots, p_n{:}z_n) \succeq (p_1{:}z_1', p_2{:}z_2', \ldots, p_n{:}z_n') \tag{7}$$

Monotonicity applies for all types of acts. The sequential acts on the left hand side of (5), (6) and (7) dominate those on the right hand side. This assumption is a weak dominance requirement. In particular, if $z_i \sim z_i'$, the substitution of a sub-tree z_i by z_i' yields an indifferent tree.

Now we turn to the reversal of order property. Let's take the stock of the situation. There are three variables R (roulette), I (ignorance) and H (horse race). Concerning reversibility, we have to examine three pairs of variables: (I, R), (H, R) and (H, I). In [8] it has been shown that (I, R) are not reversible. In the light of Anscombe-Aumann's result [1], the adoption of complete reversal of order for (H, R) would imply that H must be probabilistic. So if the nature of H is non-probabilistic, we have to reject reversibility for (R, H). For the last pair (H, I) we'll impose complete reversibility.

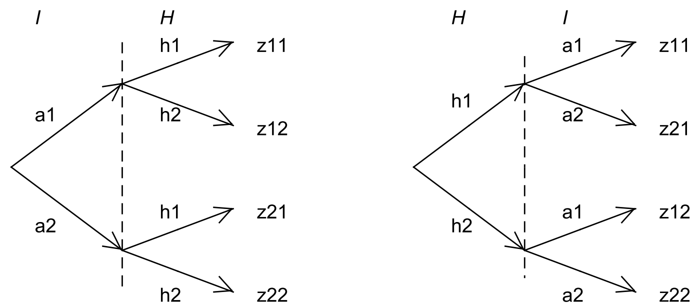

Figure 1: Reversal of order between I and H.

Assumption 3 (HI-Reversal of order (RHI)).
Suppose for HI-act $f(a_i, h_j) = z_{ij}$ where $a_i \in \Omega_I$ and $h_j \in \Omega_H$ then

$$\begin{Bmatrix} [z_{11}, z_{12}, \ldots, z_{1n}] \\ [z_{21}, z_{22}, \ldots, z_{2n}] \\ \cdots \\ [z_{k1}, z_{k2}, \ldots, z_{kn}] \end{Bmatrix} \sim \begin{bmatrix} \{z_{11}, z_{21}, \ldots, z_{k1}\} \\ \{z_{12}, z_{22}, \ldots, z_{k2}\} \\ \cdots \\ \{z_{1n}, z_{2n}, \ldots, z_{kn}\} \end{bmatrix} \tag{8}$$

This assumption regulates two sequential versions of the same functional act. On the left is a "bag of H-acts" (for ordering $I \lhd H$) while on the right is an "H-act on bags of prizes" (for ordering $H \lhd I$). Fig. 1 is an example. In the matrix notation (8), the reversal of the order between H and I is obtained by matrix transpose.

Suppose $\{A_i\}_{i=1}^2$ and $\{B_j\}_{j=1}^2$ are two partitions of Ω_H; A_iB_j denotes the conjunction of A_i and B_j, (RHI) has the following form in the rule notation:

$$\begin{Bmatrix} [A_1 \mapsto x_1, A_2 \mapsto x_2] \\ [B_1 \mapsto y_1, B_2 \mapsto y_2] \end{Bmatrix} \sim \begin{bmatrix} A_1B_1 \mapsto \{x_1, y_1\} \\ A_1B_2 \mapsto \{x_1, y_2\} \\ A_2B_1 \mapsto \{x_2, y_1\} \\ A_2B_2 \mapsto \{x_2, y_2\} \end{bmatrix}. \tag{9}$$

The indifference in (9) is intuitive. The left side of (9) conveys the idea that the individual is ignorant over the question which of two H-acts obtains. For example, if $s \in A_1B_2$ occurs then the individual has no idea which of x_1 or y_2 would be the prize. The right hand side of (9) exactly describes that situation.

Definition 1. *Suppose $A \subseteq \Omega_H$, τ is the characteristic value and $x, y, z \in \mathcal{O}$, the binary H-acts of the forms A_0^x where $x \leq \tau$, A_1^y where $y \geq \tau$ or A_τ^z are called* canonical acts *and A is called the* base event *of the canonical act. A_y^x is the compact notation for $[A \mapsto x, \bar{A} \mapsto y]$.*

The name "canonical" comes from the fact that those acts have simplest structure. They have only two branches with prizes that are on the same side of τ and at least one prize is a special value: either $0, 1$ or τ. Moreover, as we'll see later (Theorem 4) that any H-act can be decomposed into a collection of these simple acts. The set of canonical acts with base A has three parts:

$$\mathcal{H}_0^A \stackrel{\text{def}}{=} \{A_0^x |\ x \leq \tau\}, \mathcal{H}_1^A \stackrel{\text{def}}{=} \{A_1^x |\ \tau \leq x\} \text{ and } \mathcal{H}_\tau^A \stackrel{\text{def}}{=} \{A_\tau^x |\ 0 \leq x \leq 1\}.$$

Despite the fact that it is not possible to have reversibility between H and R, we still want to have this property applies in more restricted scope, namely, holds for canonical acts only.

Assumption 4 (HR-Partial reversal of order (PRHR)).
For any event $A \subseteq \Omega_H$, probability vector $(p_i)_1^k$, and $A_z^{x_i} \in \mathcal{H}_z^A$ for $1 \leq i \leq k$ and either $z = 0$ or $z = \tau$ or $z = 1$

$$(p_1 : A_z^{x_1}, p_2 : A_z^{x_2}, \ldots, p_k : A_z^{x_k}) \sim [A \mapsto (p_1 : x_1, \ldots, p_k : x_k), \bar{A} \mapsto z] \tag{10}$$

The reversal is *partial* because of its limited scope i.e. it applies only to canonical acts that have the same prizes on one of two branches. The simple binary structure of the canonical acts justifies the assumption. On the left of (10) is a linear mixture of elements of $\mathcal{H}_z^A$ associated with order $R \lhd H$. The canonical acts in the mixture are based on the same event and have the same prize on one of their two branches (Fig. 2). The right hand side explicitly specifies the element in $\mathcal{H}_z^A$ that is indifferent to the mixture. This element is associated with order $H \lhd R$. Since all x_i are on the same side of τ as z then so must be their linear mixture.

4.2. Characterization of uncertainty for H

For the rest of the paper we assume that the mother preference relation satisfies all four assumptions (OD), (M), (RHI) and (PRHR). First, we prove a lemma about the decomposition of an arbitrary binary H-act into a bag of canonical acts.

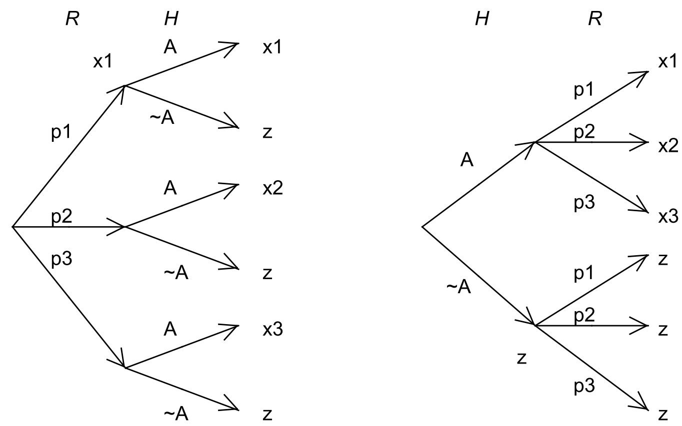

Figure 2: Partial reversal of order between R and H ($z = 0$, $z = \tau$ or $z = 1$).

Lemma 1. *Suppose* $A \subseteq \Omega_H$
(i) If $x, y \leq \tau$ *then* $A^y_x \sim \{A^y_0, \bar{A}^x_0\}$.
(ii) If $x, y \geq \tau$ *then* $A^y_x \sim \{A^y_1, \bar{A}^x_1\}$
(iii) If $x \leq \tau$ *then* $x \sim \{A^x_0, \bar{A}^x_0\}$.
(iv) If $y \geq \tau$ *then* $y \sim \{A^y_1, \bar{A}^y_1\}$.

Proof: We show (i). Application of (RHI) and (M) yields $\{A^y_0, \bar{A}^x_0\} \sim [A \mapsto \{y, 0\}, \bar{A} \mapsto \{x, 0\}] \sim [A \mapsto y, \bar{A} \mapsto x] = A^y_x$. The first indifference in the chain is due to (RHI) the second is due to (M) applying for $\{y, 0\} \sim_I y$ and $\{x, 0\} \sim_I x$ because of premise $x, y \leq \tau$. For (ii), $\{A^y_1, \bar{A}^x_1\} \sim [A \mapsto \{y, 1\}, \bar{A} \mapsto \{x, 1\}] \sim [A \mapsto y, \bar{A} \mapsto x] = A^y_x$. ($iii$) is obtained from ($i$) and ($iv$) from ($ii$) by setting $x = y$.■

The following theorem generalizes Lemma 1 that any H-act can be decomposed into a bag of canonical acts. As a result, finding CE of an H-act is reduced to determining CEs of canonical acts.

Theorem 4. *For any H-act $f = [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_m \mapsto x_m]$ where $\{A_i\}_{i=1}^m$ is a partition of Ω_H there exists a collection of canonical acts g_i, $1 \leq i \leq \ell$, such that $f \sim \{g_1, g_2, \ldots, g_\ell\}$.*

Proof: Without losing generality, we can assume that x_i are in decreasing order i.e., $x_1 > x_2 > \ldots > x_m$. There are three possibilities on the relative position of τ wrt x_1 and x_m: (i) $\tau \geq x_1$, (ii) $x_m > \tau$ and (iii) $x_1 \geq \tau \geq x_m$.

(i) Suppose $\tau \geq x_1$. All the prizes of f are worse than the characteristic value. First, consider the bag of two canonical acts $\{g_1, g_2\} = \{[B_1 \mapsto x_1, \bar{B}_1 \mapsto 0], [B_2 \mapsto x_2, \bar{B}_2 \mapsto 0]\}$. By (RHI), the bag is indifferent to $[B_1B_2 \mapsto \{x_1, x_2\}, B_1\bar{B}_2 \mapsto \{x_1, 0\}, \bar{B}_1B_2 \mapsto \{x_2, 0\}, \bar{B}_1\bar{B}_2 \mapsto \{0, 0\}]$ where AB denotes the intersection of A and B. Because $\tau \geq x_1 > x_2 \geq 0$ by assumption 1, we have $\{x_1, x_2\} \sim x_1$, $\{x_1, 0\} \sim x_1$, $\{x_2, 0\} \sim x_2$ and $\{0, 0\} \sim 0$. By (M) $\{[B_1 \mapsto x_1, \bar{B}_1 \mapsto 0], [B_2 \mapsto x_2, \bar{B}_2 \mapsto 0]\} \sim [B_1 \mapsto x_1, \bar{B}_1B_2 \mapsto x_2, \bar{B}_1\bar{B}_2 \mapsto 0]$. Thus, a bag of two canonical acts is indifferent to an act of three branches. This argument extends for the case of m canonical acts. It shows that $\{[B_i \mapsto x_i, \bar{B}_i \mapsto 0]\}_{i=1}^m \sim [B_1 \mapsto x_1, \bar{B}_1B_2 \mapsto x_2, \ldots, \bar{B}_1\bar{B}_2, \ldots \bar{B}_{i-1}B_i \mapsto x_i, \ldots \bar{B}_1\bar{B}_2, \ldots \bar{B}_m \mapsto 0]$. Thus the proof of the theorem reduces to showing that for a given act $[A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_m \mapsto x_m, A_{m+1} \mapsto 0]$ there is a collection of sets $B_1, B_2, \ldots B_m \subseteq \Omega_H$ that satisfies the following system of equations:

$$\begin{aligned}
&B_1 &&= A_1 \\
&\bar{B}_1B_2 &&= A_2 \\
&\ldots \\
&\bar{B}_1\bar{B}_2 \ldots \bar{B}_{i-1}B_i &&= A_i \\
&\ldots \\
&\bar{B}_1\bar{B}_2 \ldots \bar{B}_{m-1}B_m &&= A_m \\
&\bar{B}_1\bar{B}_2 \ldots \bar{B}_{m-1}\bar{B}_m &&= A_{m+1}
\end{aligned} \tag{11}$$

In fact, the solutions have the form

$$\begin{aligned}
&B_1 = A_1 \\
&B_2 = A_2 \cup C_1 \text{ for } C_1 \subseteq A_1 \\
&\ldots \\
&B_i = A_i \cup C_{i-1} \text{ for } C_{i-1} \subseteq A_1 \cup A_2 \ldots \cup A_{i-1} \\
&\ldots \\
&B_m = A_m \cup C_{m-1} \text{ for } C_{m-1} \subseteq A_1 \ldots \cup A_{m-1}
\end{aligned} \tag{12}$$

Of particular interest is the solution where $C_{i-1} = A_1 \cup A_2 \ldots \cup A_{i-1}$:

$$B_i = A_1 \cup A_2 \ldots \cup A_{i-1} \cup A_i. \tag{13}$$

(ii) Suppose $x_m \geq \tau$ i.e., all the prizes are better than the characteristic value. A similar argument shows that for any collection of $D_j \subseteq \Omega_H$

$$\left\{ \begin{array}{l} \left[D_1 \mapsto x_1, \bar{D}_1 \mapsto 1\right], \\ \left[D_2 \mapsto x_2, \bar{D}_2 \mapsto 1\right], \\ \ldots \\ \left[D_m \mapsto x_m, \bar{D}_m \mapsto 1\right] \end{array} \right\} \sim \left[\begin{array}{l} D_m \mapsto x_m, \\ \bar{D}_m D_{m-1} \mapsto x_{m-1}, \\ \ldots \\ \bar{D}_m \bar{D}_{m-1} \ldots D_1 \mapsto x_1, \\ \bar{D}_m \bar{D}_{m-1} \ldots \overline{D}_1 \mapsto 1 \end{array} \right]$$

So for any $[A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_m \mapsto x_m]$, the decomposition into canonical acts is satisfied by the following collection of D_i:

$$\begin{array}{l} D_m = A_m \\ D_{m-1} = A_{m-1} \cup E_m \text{ for } E_m \subseteq A_m \\ \ldots \\ D_i = A_i \cup E_{i+1} \text{ for } E_{i+1} \subseteq A_m \cup \ldots \cup A_{i+1} \\ \ldots \\ D_1 = A_1 \cup E_2 \text{ for } E_2 \subseteq A_m \cup A_{m-1} \ldots \cup A_2 \end{array} \tag{14}$$

Of particular interest is the solution where $E_i = A_m \cup \ldots \cup A_i$

$$D_i = A_m \cup \ldots \cup A_{i+1} \cup A_i \tag{15}$$

(iii) Suppose $x_1 > \tau > x_m$ i.e., the prize range includes the characteristic value. There exists an index $1 \leq i < m$ such that $x_i \geq \tau > x_{i+1}$. We will show that $f = [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_m \mapsto x_m]$ is indifferent to another act whose prizes lie entirely on the same side of τ. Thus the decompositions given in cases (i) and (ii) apply.

Due to (M), $x_1 \succeq f \succeq x_m$. Suppose $f \succeq \tau$, consider bag $\{f, x_1\}$. On the one hand, as both f and x_1 are preferable to τ so by assumption 1, $\{f, x_1\} \sim f$ because f is the less preferable element in the bag. On the other hand, by (RHI)

$$\begin{aligned} \{f, x_1\} &\sim [A_1 \mapsto x_1, A_2 \mapsto \{x_2, x_1\}, \ldots, A_m \mapsto \{x_m, x_1\}] && (16) \\ &\sim [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_i \mapsto x_i, A_{i+1} \mapsto \tau, \ldots, A_m \mapsto \tau] && (17) \\ &= [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_i \mapsto x_i, A_{i+1} \cup \ldots \cup A_m \mapsto \tau] && (18) \\ &= g_u \end{aligned}$$

By the transitivity $f \sim g_u$. Thus the decomposition of g_u the in case (ii) applies for f.

If $\tau \succeq f$ then $\{f, x_m\} \sim f$. A similar argument shows that

$$f \sim \{f, x_m\} \sim [A_1 \cup \ldots \cup A_i \mapsto \tau, A_{i+1} \mapsto x_{i+1}, \ldots, A_m \mapsto x_m] = g_d \tag{19}$$

As all prizes of g_d are below τ, the decomposition in case (i) is applicable.■

We now turn to examining canonical acts and the roulette variable R. Let's consider the lotteries taking values in the set of canonical acts of the same base $\mathcal{H}_z^A$: $(p_1 : A_z^{x_1}, p_2 : A_z^{x_2}, \ldots, p_k : A_z^{x_k})$ with $z = 0, \tau, 1$. (PRHR) assumption stipulates reversibility between H and R for this type of acts.

Let's look at binary (two-branch) lotteries of the form $(p{:}\tau, 1-p{:}0)$. By the assumption 1 (OD), for any $0 \leq x \leq \tau$, there is a probability value $\lambda(x)$ such that $x \sim (\lambda(x){:}\tau, 1-\lambda(x){:}0)$. Given the utility function representing $\succeq_R$, the values $\lambda(x)$ is completely determined. Similarly, for any $\tau \leq y \leq 1$, there is unique value $\rho(y)$ that validates $y \sim (\rho(y){:}\tau, 1-\rho(y){:}1)$. Technically $\lambda(x)$ is defined only for $x \leq \tau$ and $\rho(x)$ only for $x \geq \tau$. We can extend the definitions $\lambda(x), \rho(x)$ to entire range of $\mathcal{O}$ with the conventions $\lambda(x) = 1$ for $x > \tau$ and $\rho(y) = 1$ for $y < \tau$. The explicit expressions are given as follows.

Definition 2. *Suppose u is the utility function representing $\succeq_R$ such that $u(0) = 0$ and $u(1) = 1$.*

$$\lambda(x) \stackrel{\text{def}}{=} \begin{cases} \frac{u(x)}{u(\tau)} & \text{for } x < \tau \\ 1 & \text{for } x \geq \tau \end{cases} \quad \text{and } \rho(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } x \leq \tau \\ \frac{1-u(x)}{1-u(\tau)} & \text{for } x > \tau \end{cases} \tag{20}$$

For any $0 \leq x \leq 1$, either $\lambda(x) = 1$ or $\rho(x) = 1$. Also $\lambda(\tau) = \rho(\tau) = 1$. For the values $\tau = 0$ or $\tau = 1$ either $\lambda(x)$ or $\rho(y)$ are not defined. A numerical example is given in Section 6.2.

The following theorem shows that each subset of $A \subseteq \Omega_H$ is associated with a number determined via the indifference between A-based canonical acts and binary lotteries.

Theorem 5. *For each $A \subseteq \Omega_H$ there is a unique value $\pi(A)$ such that*

$$A_0^x \sim (\lambda(x)\pi(A){:}\tau, 1-\lambda(x)\pi(A){:}0) \quad \textit{for } x \leq \tau \tag{21}$$

$$A_1^y \sim (\rho(y)\pi(A){:}\tau, 1-\rho(y)\pi(A){:}1) \quad \textit{for } \tau \leq y \tag{22}$$

where $\lambda(x)$, $\rho(y)$ are defined in (20).

To see the significance of this theorem note that (21) and (22) establish two cases of indifference mapping between canonical H-acts and binary lotteries (R-act). This indifference is a bridge to the CE of the canonical act because the CE of the corresponding lottery is determined via the linear utility function u representing preference over lotteries. Moreover, the prizes of the binary lotteries are fixed ($0, \tau$ for $x \leq \tau$ and $\tau, 1$ for $y \geq \tau$). The only parameter that accounts for the variation of the CE due to variation of x and A is the product $\lambda(x)\pi(A)$. Thus, $\lambda(x)$ and $\rho(y)$ encapsulate the effect of prizes x and y while $\pi(A)$ captures the effect of the base event. The total combined effect of the prize and the base event is the product of the component effects. Note also that the same value $\pi(A)$ appears in both cases of indifference for A_0^x and A_1^y. We call $\pi(A)$ the *H-measure* of A. In particular, $\pi(\Omega_H) = 1$ and $\pi(\emptyset) = 0$. A numerical example showing extraction of H-measure from revealed preference is given in Section 6.3.

Proof: Suppose that u is the utility function representing $\succeq_R$ such that $u(0) = 0$, $u(1) = 1$; and t is the probability value that verifies the indifference $\tau \sim (t{:}1, 1-t{:}0)$. Because $0 < \tau < 1$ and $u(\tau) = t$, $0 < t < 1$.

For $x \leq \tau \leq y$, we have $x \sim (\lambda(x) : \tau, 1 - \lambda(x) : 0)$ and $y \sim (\rho(y) : \tau, 1 - \rho(y){:}1)$. For event $A \subseteq \Omega_H$, denote by a_A and b_A the CEs of A_0^τ and A_1^τ respectively i.e., $A_0^\tau \sim a_A$ and $A_1^\tau \sim b_A$. Clearly, $0 \leq a_A \leq \tau \leq b_A \leq 1$. Denote by $\pi(A)$ and $\beta(A)$ the probabilities that verify the indifferences

$$(\pi(A){:}\tau, 1-\pi(A){:}0) \sim_R a_A \text{ and } (\beta(A){:}\tau, 1-\beta(A){:}1) \sim_R b_A.$$

By transitivity of $\succeq$ we have the indifferences between canonical H-acts and lotteries:

$$A_0^\tau \sim (\pi(A){:}\tau, 1-\pi(A){:}0) \quad \text{and} \quad A_1^\tau \sim (\beta(A){:}\tau, 1-\beta(A){:}1) \tag{23}$$

Let's consider the following indifference chains:

$$\begin{aligned}
A_0^x &\sim [A \mapsto (\lambda(x){:}\tau, 1-\lambda(x){:}0), \bar{A} \mapsto 0] \sim (\lambda(x){:}A_0^\tau, 1-\lambda(x){:}0) \\
&\sim (\lambda(x){:}(\pi(A){:}\tau, 1-\pi(A){:}0), 1-\lambda(x){:}0) \\
&\sim (\lambda(x)\pi(A){:}\tau, 1-\lambda(x)\pi(A){:}0) \qquad (24) \\
A_1^y &\sim [A \mapsto (\rho(y){:}\tau, 1-\rho(y){:}1), \bar{A} \mapsto 1] \sim (\rho(y){:}A_1^\tau, 1-\rho(y){:}1) \\
&\sim (\rho(y){:}(\beta(A){:}\tau, 1-\beta(A){:}1), 1-\rho(y){:}1) \\
&\sim (\rho(y)\beta(A){:}\tau, 1-\rho(y)\beta(A){:}1) \qquad (25)
\end{aligned}$$

The first indifference in the chain (24) is due to (M) where x is substituted by an indifferent lottery. The next indifference is due to (PRHR). The third is again due to (M) where A_0^τ is substituted by an indifferent lottery

$(\pi(A)\!:\!\tau, 1-\pi(A)\!:\!0)$. The final step where the indifference is obtained by collapsing two-stage lottery into a one-stage lottery. The indifference is due to assumption that preference on lotteries is represented by a linear utility function. The reasoning behind (25) is similar. Note that (25) is different from (22). To prove the latter from the former it is necessary to show that $\pi(A) = \beta(A)$.

On the one hand, let's consider the indifference chain: $(t\!:\!b_A, 1-t\!:\!a_A) \sim (t:A_1^\tau, 1-t:A_0^\tau) \sim [A \mapsto (t:\tau, 1-t:\tau), \bar{A} \mapsto (t:1, 1-t:0)] \sim \tau$. Thus, $(t\!:\!b_A, 1-t\!:\!a_A) \sim_R \tau$. Hence,

$$u((t\!:\!b_A, 1-t\!:\!a_A)) = u(\tau) = t. \tag{26}$$

On the other hand, from $(\pi(A)\!:\!\tau, 1-\pi(A)\!:\!0) \sim a_A$ and $(\beta(A)\!:\!\tau, 1-\beta(A)\!:\!1) \sim b_A$, it follows that $u(a_A) = \pi(A)u(\tau) = t.\pi(A)$ and $u(b_A) = 1-(1-t)\beta(A)$.

$$u((t\!:\!b_A, 1-t\!:\!a_A)) = t(1-(1-t)\beta(A)) + t(1-t)\pi(A) \tag{27}$$

Comparing (26) and (27) we have $1-(1-t)\beta(A) + (1-t).\pi(A) = 1$ or $(1-t)(\pi(A)-\beta(A)) = 0$. Because $t < 1$, it follows that $\pi(A) = \beta(A)$. Thus, (22) is proved.■

Although $\lambda(x)$ and $\rho(y)$ are probabilities, $\pi(A)$ is not. The following theorem shows that π behaves like a possibility measure.

Theorem 6. *For measure π defined in theorem 5*
(i) For $A \subseteq \Omega_H$, either $\pi(A) = 1$ or $\pi(\bar{A}) = 1$.
(ii) For $A, B \subseteq \Omega_H$, $\pi(A \cup B) = \max(\pi(A), \pi(B))$

Proof: We prove an equivalent statement of (ii): if $\pi(A) \geq \pi(B)$ then $\pi(A) = \pi(A \cup B)$. Let's consider bag $\{A_0^\tau, B_0^\tau\}$. On the one hand, because both elements of the bag are not preferable than τ, the bag is indifferent to the preferable element between the two. From the premise $\pi(A) \geq \pi(B)$, it follows that $(\pi(A):\tau, 1-\pi(A):0) \succeq (\pi(B):\tau, 1-\pi(B):0)$. By theorem 5, it follows that $A_0^\tau \succeq B_0^\tau$. So we have the indifference $\{A_0^\tau, B_0^\tau\} \sim A_0^\tau$. On the other hand, applying the reversal of order between I and H (RHI) yields $\{A_0^\tau, B_0^\tau\} \sim [AB \mapsto \tau, A\bar{B} \mapsto \{\tau, 0\}, \bar{A}B \mapsto \{\tau, 0\}, \bar{A}\bar{B} \mapsto 0] \sim [A \cup B \mapsto \tau, \bar{A}\bar{B} \mapsto 0]$ where AB denotes the intersection of A and B. Therefore, $A_0^\tau \sim [A \cup B \mapsto \tau, \bar{A}\bar{B} \mapsto 0]$. Thus, $\pi(A) = \pi(A \cup B)$. (i) directly follows from (ii) when $B = \bar{A}$.■

We have shown that the uncertainty that regulates the outcome of H satisfies the axioms of possibility theory. As we have seen, the numerical aspect of π is based on its partial reversibility wrt the probabilistic variable R and the max-composition rule is due to its complete reversibility wrt the ignorant variable I.

5. The certainty equivalent of H-act

This section concerns with finding the certainty equivalents of H-acts. We first present a theorem that establishes indifference between an H-act and a special binary lottery. This section concludes with an algorithm to calculate the certainty equivalents.

Theorem 7. *Suppose $f = [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_m \mapsto x_m]$ is an H-act where $\{A_i\}_{i=1}^m$ is a partition of Ω_H, $x_i \in \mathcal{O}$ and λ^*, ρ^* are defined by:*

$$\lambda^* \stackrel{\text{def}}{=} \max_{0 \le i \le m} \{\pi(A_i)\lambda(x_i)\} \text{ and } \rho^* \stackrel{\text{def}}{=} \max_{0 \le i \le m} \{\pi(A_i)\rho(x_i)\}. \tag{28}$$

We have $\max(\lambda^*, \rho^*) = 1$ *and the following contingent indifference*

$$f \sim \begin{cases} (\lambda^* {:} \tau, 1 - \lambda^* {:} 0) & \text{if } \lambda^* < 1,\ \rho^* = 1 \\ (\rho^* {:} \tau, 1 - \rho^* {:} 1) & \text{if } \lambda^* = 1,\ \rho^* < 1 \\ \tau & \text{if } \lambda^* = \rho^* = 1 \end{cases}. \tag{29}$$

Note that λ^* and ρ^* are the maxima over the set of H-measures for A_i adjusted by $\lambda(x_i)$ and $\rho(x_i)$ that are defined in Definition 2. Eq. (29) has three cases of indifference between an H-act and a lottery.

Proof: The proof uses the results and proofs of theorems 4 and 5. Assume without loss of generality that $(x_i)_{i=1}^m$ are in decreasing order and k is the largest index such that $x_k \ge \tau$. Let's consider two bags containing f: $\{f, x_1\}$ and $\{f, x_m\}$ where x_1 (x_m) is the most (least) preferable prize. Using identity $x = [A_1 \mapsto x, A_2 \mapsto x, \ldots, A_m \mapsto x]$ and applying the reversal of order (RHI) and (M), we have

$$\{f, x_1\} \sim [A_1 \mapsto x_1, A_2 \mapsto x_2, \ldots, A_k \mapsto x_i, A_{k+1} \ldots \cup A_m \mapsto \tau] = f_u \tag{30}$$

$$\{f, x_m\} \sim [A_1 \cup \ldots \cup A_k \mapsto \tau, A_{k+1} \mapsto x_{k+1}, \ldots A_m \mapsto x_m] = f_d \tag{31}$$

Applying theorem 4 for f_d, we have

$$f_d \sim \left\{ \begin{array}{l} [A_1 \mapsto \tau, \bar{A}_1 \mapsto 0] \\ \dots \\ [A_k \mapsto \tau, \bar{A}_k \mapsto 0] \\ [A_{k+1} \mapsto x_{k+1}, \bar{A}_{k+1} \mapsto 0] \\ \dots \\ [A_m \mapsto x_m, \bar{A}_m \mapsto 0] \end{array} \right\} \tag{32}$$

Because each member of the bag is not preferable to τ, the entire bag is indifferent to the most preferable member. Each member, by theorem 5, is indifferent to a lottery.

$$\begin{array}{l} [A_i \mapsto \tau, \bar{A}_i \mapsto 0] \sim (\pi(A_i)\lambda(\tau){:}\tau, 1 - \pi(A_i){:}0) \ \text{ for } \ i \leq k \\ [A_i \mapsto x_i, \bar{A}_i \mapsto 0] \sim (\pi(A_i)\lambda(x_i){:}\tau, 1 - \pi(A_i)\lambda(x_i){:}0) \ \text{ for } \ i > k \end{array}$$

So, the most preferable among those lotteries is the one with maximal probability of getting τ i.e. $\lambda^* = \max_i \pi(A_i)\lambda(x_i)$. Following similar reasoning for f_u we can show that

$$f_d \sim (\lambda^*{:}\tau, 1 - \lambda^*{:}0) \text{ and } f_u \sim (\rho^*{:}\tau, 1 - \rho^*{:}1) \tag{33}$$

where λ^* and ρ^* are defined by (28).

Next we show that $\max(\lambda^*, \rho^*) = 1$. Note that by Definition 2 we have $\max(\lambda(x), \rho(()x)) = 1$, so $\max(\lambda^*, \rho^*) = \max\{\pi(A_k) | 1 \leq k \leq m\}$. Since A_i forms a partition of Ω_H, by theorem 6, $\max\{\pi(A_i) | 1 \leq i \leq m\} = 1$. Thus, for any act f there are only three possibilities: (a) $\lambda^* < 1, \rho^* = 1$; (b) $\lambda^* = 1, \rho^* < 1$ and (c) $\lambda^* = \rho^* = 1$. We consider each case as follows.

(a) If $\lambda^* < 1$ then $\tau \succ (\lambda^*{:}\tau, 1 - \lambda^*{:}0) \sim f_d \sim \{f, x_m\}$. From $f \succeq x_m$, it follows that $f \sim (\lambda^*{:}\tau, 1 - \lambda^*{:}0)$.

(b) If $\rho^* < 1$ then $\{f, x_1\} \sim f_u \sim (\rho^*{:}\tau, 1 - \rho^*{:}1) \succ \tau$. Because $x_1 \succeq f$, $f \sim (\rho^*{:}\tau, 1 - \rho^*{:}1)$.

(c) If $\lambda^* = \rho^* = 1$ then $\{f, x_1\} \sim \{f, x_m\} \sim \tau$. So $f \sim \tau$.

This concludes the proof.■

If we calculate the expected utility for the lotteries on the right hand side of (29) then that value can be viewed as the "utility" for the horse act. Recall that u is a linear utility function given in Definition 2, abusing notation slightly we can extend it for the horse acts too. We can combine three cases of (29) into one expression.

$$u(f) = \lambda^* u(\tau) + (1 - \rho^*)(1 - u(\tau)) \tag{34}$$
$$= \max_{0 \le i \le m} \{\pi(A_i)\lambda(x_i)\} u(\tau) + (1 - \max_{0 \le i \le m} \{\pi(A_i)\rho(x_i)\}) (1 - u(\tau)) \tag{35}$$

(35) shows how the utility of H-act factors in four sources of information: uncertainty of the states (π), prizes (x_i), risk attitude (u) and finally, the tolerance for ignorance (τ).

To conclude this section, we have an algorithm to find the CE of H-act suggested by Theorem 7.

1. Determine values $\lambda(x_i), \rho(x_i)$ via Definition 2 using utility function u representing preference under risk.
2. Determine H-measures $\pi(A_i)$ for A_i via (21) and (22) using the indifference between canonical acts and binary lotteries.
3. Determine the utility and the CE based on Eq. (35)

6. Investment under heterogeneous uncertainty: An example

In this section we provide a running example to illustrate the issues discussed earlier in the paper. This example extends the one given in [8] and cited in Section 3. An investor considers investments whose returns depend on the outcomes of three uncertain variables of different types. Variable R stands for the proposition about 2017 coffee crop in a country (e.g., Brazil). The possibles outcomes are "poor" (p), "normal" (n) and "bumper" (b). The probability distribution $(0.34, 0.2, 0.46)$ is estimated on the basis of reliable historical data as well as careful meteorological modeling. Variable I stands for the prospect of peace settlement in a conflict (e.g., Afghanistan) before January 1, 2018. This variable has two possible outcomes: settlement (s) and no settlement (s̃). The experts whom the investor consults on this question do not agree with each other and give conflicting advices. Their track records on this question are poor. The investor concludes that the experts do not have a firm grip of the driving forces of politics in that part of the world. She considers I an ignorant variable. The third variable H concerning the possible future of a country (e.g., Greece). This variable has four possible outcomes $\Omega_H = \{h_1, h_2, h_3, h_4\}$ where h_1 stands for "by January 1, 2018 Greece will remain in Eurozone and the European Union (EU)"; h_2 -

"Greece will stay in EU but outside Eurozone"; h_3 - "Greece will be outside of both EU and Eurozone"; and h_4 - "Greece will join the economic block led by Russia".[5] There is no credible argument to suggest any degree of dependency between variables. The investor has some information about H but she does not feel comfortable to commit to a probability distribution. Moreover, she is not sure which formalism she should use to express the uncertainty. She feels that she would make different choice (today) if she knew the outcome of R would be realized before the outcome of H than vice versa. However, she feels indifferent about the order between the two geopolitical factors. That is, she would act the same way no matter which question will be resolved first: peace in Afghanistan or the future of Greece in EU.

Table 2 has contingent payoffs of an investment f. The left part of the table has the returns when $R = p$. The column labels tell the value of H and row labels - the value of I. Instead of stating the returns on investments in percentages, we assume that the returns are normalized to be in range $[0, 1]$.[6] For example $f(R = p, I = s, H = h_1) = 0.4$ means that in case Brazil has a poor coffee crop, peace is established in Afghanistan and Greece stays in both EU and Eurozone, the investment yields 0.4 unit of return. The other parts of the table have the returns of investment when $R = n$ or $R = b$.

p	h_1	h_2	h_3	h_4	n	h_1	h_2	h_3	h_4	b	h_1	h_2	h_3	h_4
s	.4	.3	.1	.2	s	.3	.5	.2	.4	s	.5	.2	.7	.1
$\tilde{s}$	0	.7	.1	.6	$\tilde{s}$	.3	.7	.1	1	$\tilde{s}$	.9	.7	.1	0

Table 2: Normalized payoff of an investment.

6.1. Attitudes toward risk and ignorance

Assume that individual's risk attitude is described by a power utility function (CRRA) [17] $u(x) \propto x^r$ with $r = 0.4$ that is normalized so that $u(0) = 0$ and $u(1) = 1$ (Fig. 3) and her characteristic value under ignorance $\tau = 0.2$. For example, the individual calculates the prices for the lotteries based on Brazilian coffee crop by linear utility function u. To compare acts

[5] This example was formulated in May of 2015 and was inspired by the current events. Readers can refer to the front pages of newspapers such as *the Financial Times* during the period for the context.

[6] A simple normalization procedure is presented in [8].

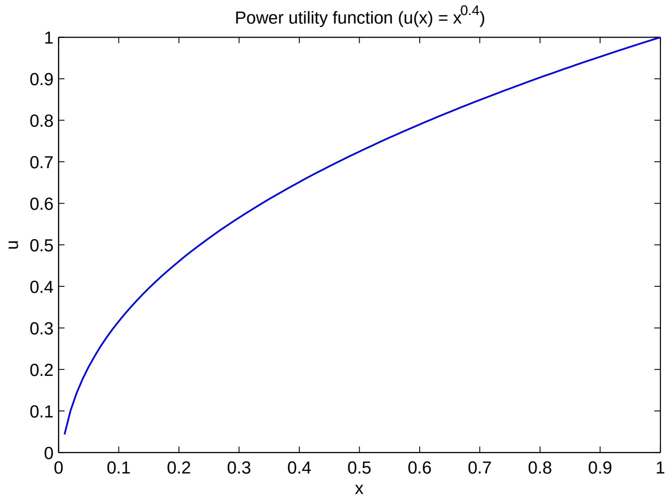

Figure 3: Power utility function.

x	u	$\lambda(x)$	$\rho(x)$	x	u	$\lambda(x)$	$\rho(x)$
0.0100	0.0436	0.0947	1.0000	0.5000	0.7248	1.0000	0.5101
0.1000	0.3159	0.6861	1.0000	0.6000	0.7900	1.0000	0.3893
0.2000	0.4605	1.0000	1.0000	0.7000	0.8489	1.0000	0.2801
0.3000	0.5656	1.0000	0.8051	0.8000	0.9030	1.0000	0.1799
0.4000	0.6513	1.0000	0.6464	0.9000	0.9531	1.0000	0.0869

Table 3: Fragment of mapping from x to $\lambda(x), \rho(x)$.

that depend on the peace settlement the individual uses the τ-anchor utility function with $\tau = 0.2$. These two pieces of information are sufficient to characterize the individual behavior under uncertainty comprising of ignorance and probability as discussed in [8].

6.2. Determining $\lambda(x), \rho(x)$

Table 3 has $\lambda(x), \rho(x)$ for some values of x calculated by Eq. (20).

6.3. Extracting subjective uncertainty for H

Suppose the individual reveals the indifferences in the left column of Table 4. For example, H-act $[h_1 \mapsto 0.2, \{h_2, h_3, h_4\} \mapsto 0]$ is a gamble that pays 0.2 unit if "Greece remains in both Eurozone and EU by January 1, 2018" and zero unit otherwise. Observation $[h_1 \mapsto 0.2, \{h_2, h_3, h_4\} \mapsto 0] \sim 0.15$ means

User's indifference	Calculated π
$[h_1 \mapsto 0.2, \{h_2, h_3, h_4\} \mapsto 0] \sim 0.15$	$\pi(h_1) = 0.8591$
$[h_2 \mapsto 0.2, \{h_1, h_3, h_4\} \mapsto 0] \sim 0.20$	$\pi(h_2) = 1$
$[h_3 \mapsto 0.3, \{h_1, h_2, h_4\} \mapsto 1] \sim 0.40$	$\pi(h_3) = 0.8029$
$[h_4 \mapsto 0.5, \{h_2, h_3, h_1\} \mapsto 1] \sim 0.70$	$\pi(h_4) = 0.5491$

Table 4: H-measure extracted from user preference.

that 0.15 unit is the level of price that the individual would be willing to buy/sell the gamble (ignoring transaction cost).

We use theorem 5 to calculate H-measures for members of Ω_H. To help with readability we rewrite eqs. (21) and (22) here.

$$A_0^x \sim (\lambda(x)\pi(A){:}\tau, 1 - \lambda(x)\pi(A){:}0) \text{ for } x \leq \tau$$
$$A_1^y \sim (\rho(y)\pi(A){:}\tau, 1 - \rho(y)\pi(A){:}1) \text{ for } \tau \leq y.$$

From observation $[h_1 \mapsto 0.2, \{h_2, h_3, h_4\} \mapsto 0] \sim 0.15$, it follows that $(\lambda(0.2)\pi(h_1){:}\tau, 1 - \lambda(0.2)\pi(h_1){:}0) \sim 0.15$. Because $\tau = 0.2$, $\pi(h_1)$ is the solution for:

$$\lambda(0.2)\pi(h_1)u(0.2) + (1 - \lambda(0.2)\pi(h_1))u(0) = u(0.15)$$

where $\lambda(0.2) = 1$, $u(0) = 0$, $u(0.15) = 0.3956$ and $u(\tau) = u(0.2) = 0.4605$. Therefore, $\pi(h_1) = u(0.15)/u(0.2) = 0.8591$. Similarly, $\pi(h_2) = u(0.2)/u(\tau) = 1$. $\pi(h_3)$ is calculated from $(\rho(0.3)\pi(h_3){:}0.2, 1 - \rho(0.3)\pi(h_3){:}1) \sim 0.4$.

$$\rho(0.3)\pi(h_3)u(0.2) + (1 - \rho(0.3)\pi(h_3))u(1) = u(0.4)$$

where $\rho(0.3) = 0.8051$, $u(0.2) = 0.4605$ and $u(0.4) = 0.6513$. Hence, $\pi(h_3) = 0.8029$. Similarly $\pi(h_4) = 0.5491$.

We just demonstrate how to determine the H-measure of an H-event from a single revealed indifference that involves a canonical act based on that event. However, theorem 5 guarantees that the value of H-measure would be the same even if it is calculated from another indifference involving a different canonical act with the same base event. For example, H-measures of A calculated from $A_0^x \sim c$ and $A_0^{x'} \sim c'$ for $x \neq x'$ must be equal. If they are not then the revealed indifferences violate one of the assumptions used in theorem 5.

The H-measure for any subset of Ω_H is determined from $\pi(h_i)$ by maximization (theorem 6).

6.4. Calculating CE for H-acts

The algorithm in Section 5 is applied for $[\{h_1, h_3\} \mapsto 0.3, \{h_2, h_4\} \mapsto 0.7]$:

(1) $\lambda(0.3) = 1;\ \rho(0.3) = 0.8051$ and $\lambda(0.7) = 1;\ \rho(0.7) = 0.2801$
(2) $\pi(\{h_1, h_3\}) = 0.8591$ and $\pi(\{h_2, h_4\}) = 1$
(3) $\lambda = \max(0.8591, 1) = 1$ and $\rho = \max(0.8051 * 0.8591, 0.2801) = 0.6917$
(4) $u^{-1}(0.6917 * u(0.2) + (1 - 0.6917)) = 0.3697$

The CE of $[\{h_1, h_3\} \mapsto 0.3, \{h_2, h_4\} \mapsto 0.7]$ is 0.3697. Thus, the individual equalizes payment of 0.3697 for sure with a gamble that pays 0.3 unit if "by January 1, 2018 Greece will remain in Eurozone and the European Union (EU)" or "Greece will be outside of both EU and Eurozone" and 0.7 unit otherwise.

6.5. Showing reversibility between I and H

This example illustrates the reversibility between I and H. Consider two H-acts: $f_1 = [\{h_1, h_3\} \mapsto 0.7, \{h_2, h_4\} \mapsto 0.2]$ and $f_2 = [h_1 \mapsto 0.5, h_2 \mapsto 0.4, h_3 \mapsto 0.7, h_4 \mapsto 1]$. We show that the bag $f = \{f_1, f_2\}$ is indifferent to the H-act on bags of prizes obtained by reversing the order between H and I. First, following the steps to calculate the CE for f_1 and f_2, we have $f_1 \sim 0.2$ and $f_2 \sim 0.4$. So, $f = \{f_1, f_2\} \sim \{0.2, 0.4\} \sim 0.2 = \tau$.

Let's consider f' obtained from f by reversing the order of variables

$$\begin{aligned} f' &= [h_1 \mapsto \{0.7, 0.5\}, h_2 \mapsto \{0.2, 0.4\}, h_3 \mapsto \{0.7, 0.7\}, h_4 \mapsto \{0.2, 1\}] \\ &\sim [h_1 \mapsto 0.5, h_2 \mapsto 0.2, h_3 \mapsto 0.7, h_4 \mapsto 0.2] \end{aligned}$$

Following the 4-step process, we have $\lambda = 1$ and $\rho = 1$. Hence $f' \sim 0.2$.

6.6. Showing partial reversibility between H and R

To illustrate this property expressed by Eq. (10), for example, we take $z = 0$ and $A = \{h_1, h_3\}$. From Table 4, $\pi(A) = 0.8591$ and $\pi(\bar{A}) = 1$. Let's consider two canonical acts $f_1 = \left[A \mapsto 0.1, \bar{A} \mapsto 0\right]$ and $f_2 = \left[A \mapsto 0.15, \bar{A} \mapsto 0\right]$. Their CEs are 0.0772 and 0.1137 respectively. We will show that the CE of a lottery on f_1, f_2 is the same as the CE of the act obtained by reversing the order of variables R and H.

The lottery on the canonical acts we consider is $g = \left(\frac{1}{3} : f_1, \frac{2}{3} : f_2\right)$. The utility $u(g) = \frac{1}{3}u(0.0772) + \frac{2}{3}u(0.1137) = 0.3171$. The CE of g is 0.1006.

Reversing the variables of g we have $g' = [A \mapsto \left(\frac{1}{3}{:}0.1, \frac{2}{3}{:}0.15\right), \bar{A} \mapsto 0]$. The expected utility of the lottery inside g' is $u\left(\left(\frac{1}{3}{:}0.1, \frac{2}{3}{:}0.15\right)\right) = 0.3691$. Its CE is 0.1320. Thus, $g' \sim [A \mapsto 0.1320, \bar{A} \mapsto 0]$. With $\lambda(0.1320) = 0.8014$ $\rho(0.1320) = 1$ and $\lambda(0) = 0$ $\rho(0) = 1$, we have $\lambda = 0.6885$ and $\rho = 1$. The CE of g' is also 0.1006.

6.7. Calculating CE of sequential acts

We now proceed to solve for the CE of the investment given in Table 2. The probability distribution for R: $Pr(p) = 0.34$, $Pr(n) = 0.2$ and $Pr(b) = 0.46$. The uncertainty about H is found in Table 4. The investor assumes the order of variables to be $R \lhd H \lhd I$ arguing that the Brazilian coffee crop will be known by the end of 2017 while the outcomes of peace settlement and membership of Greece in EU and Eurozone will only decided in 2018. We can encode the situation by the sequential act on the left of (36). Note that the H-acts are denoted in the matrix form. The RH-act on the right of (36) is obtained by replacing each bag of prizes by its CE under the assumption that $\tau = 0.2$. For example $\{.4, 0\} \sim 0.2$.

$$\begin{pmatrix} .34{:}[\ \{.4,0\},\{.3,.7\},\{.1,.1\},\{.2,.6\}\ ] \\ .20{:}[\ \{.3,.3\},\{.5,.7\},\{.2,.1\},\{.4,1\}\ ] \\ .46{:}[\ \{.5,.9\},\{.2,.7\},\{.7,.1\},\{.1,0\}\ ] \end{pmatrix} \sim \begin{pmatrix} .34{:}[\ .2,.3,.1,.2\ ] \\ .20{:}[\ .3,.5,.2,.4\ ] \\ .46{:}[\ .5,.2,.2,.1\ ] \end{pmatrix} \quad (36)$$

The next step is to find CEs for H-acts inside RH-act on the right of (36) using the 4-step procedure. Let's do so for $[.2, .3, .1, .2]$.

1. From Table 3: $\lambda(0.1) = 0.6861, \rho(0.1) = 1$; $\lambda(0.2) = 1, \rho(0.2) = 1$; $\lambda(0.3) = 1, \rho(0.3) = 0.8051$.
2. From Table 4: $\pi(h_1) = 0.8591, \pi(h_2) = 1, \pi(h_3) = 0.8029, \pi(h_4) = 0.5491$.
3. By formula (28): $\lambda = 1$, $\rho = 0.8591$.
4. The CE is 0.270 i.e., $[.2, .3, .1, .2] \sim 0.27$.

Similarly, for $[.3, .5, .2, .4]$ we have $\lambda = 1, \rho = 0.8029$. Hence, CE is 0.30. For $[.5, .2, .2, .1]$, $\lambda = 1, \rho = 1$. Hence CE is 0.2.

So the RH-act on the right hand side of (36) is indifferent to lottery $(.34{:}.27, .20{:}.30, .46{:}.20)$. Let's calculate its expected utility:

$$EU = .34 * u(.27) + .20 * u(.30) + .46 * u(.20) = .5074.$$

The CE of the lottery is $u^{-1}(.5074) = .245$. We conclude that the investment under uncertainty given in Table 2 is worth the same as the investment that has sure return of .245.

7. Discussion and related literature

As this work extends Anscombe-Aumann's multiple-variable approach used to define subjective probability, it is worthwhile to have a closer look at the two critical assumptions in AA approach. Between Monotonicity and Reversal of order assumptions, the former is well understood and widely accepted but the latter is less so. The first argument to justify the reversal of order, as AA put it, "if the prize you receive is to be determined by both a horse race and the spin of a roulette wheel, then it is immaterial whether the wheel is spun before or after the race" ([1] p.201). Essentially, the argument established the nexus between the independence between variables (the horse race and the roulette spin) and the invariance of the value of the act with respect to the order of variables. This requirement sounds reasonable but the question is whether it should hold in any circumstance.

Let's have a closer look at this argument. The notion of independence between uncertain variables is habitually reduced in practice to the notion of stochastic independence between two probabilistic variables that is characterized by a multiplicative probability condition. But when one of the involved variables is not probabilistic that reduction is not applicable for the obvious reason. A more general idea of independence is formulated in terms of conditionals, whereby the learning about a realization of one variable does not change the belief about the other (i.e. conditional belief is the same as the unconditional belief). In other words, two variables are independent if knowing the realization of one would not affect the belief about the other. The relationship between a roulette spin and a horse race in the AA description clearly fits this definition of independence. The formulation of independence by conditionals also offers the ability to account for the temporal order between variables.

Traditionally, decision problems are formulated as mappings from a state space into a reward space, hence, have only one stage. This formulation, in fact, considers only one variable even if the decision maker conceptually thinks of multiple variables. For AA acts, for example, the state space is the set of conjunctions consisting of an outcome for each spin and an outcome for the horse race. Such a formulation has its root in the matrix (strategic) form of a game in game theory. Under certain conditions, analysis of the matrix form is sufficient to solve a game even if the game is actually played in the extensive form. In a sense, AA postulate of reversal of order is a statement that confirms the traditional one-stage formulation of decision problems using the language of multiple variables. Basically, if the order between variables does not matter then the multiple variables can be collapsed into one. However, it is well known that in general the matrix form and the extensive form are not equivalent. This argument shows that the requirement of reversal of order for multiple variables is not always justified.

Another argument against the reversal of order axiom starts with the recognition that in real world situations there exists uncertainty that does not lend itself to probability theory. Suppose that an individual is completely ignorant about a particular horse race. If she adopts the reversal of order assumption, then, by the force of AA theorem, she assigns probabilities to the outcomes of the horse race. So a dilemma for the individual is between to recognize her own epistemic state of ignorance or to satisfy a normative rule of reversal of order between the horse race and roulette spin. We argue that the individual should choose the former and reject the latter. Indeed, in [8], we study decision making under uncertainty comprising two opposite states of knowledge that are complete ignorance and probability. We show that even if the variable of ignorance and the variable of probability are independent, they are not reversible in the Anscombe-Aumann's sense. In [18] (Chapter 10) Wakker also carefully discusses the issues related to the order between ambiguity and probability in AA example.

In possibility theory literature, there are few works that specifically address the question of the subjective foundation of possibility. Among those, Dubois-Prade-Smets' work [6] (DPS) is most well known because their proposal is discussed in the context of a comprehensive list of references on the subject up to 2006. We focus on the main idea of DPS approach. They argued that the probability distribution that an individual uses in making decision (revealed through betting rates) reflects the individual's epistemic belief that is not directly observable. They hypothesize that the internal epistemic belief of an individual is represented by a Dempster-Shafer belief function and she uses pignistic transformation to convert her belief function into a probability function when making decision becomes necessary. The "credal-pignistic" hypothesis was proposed by Smets and Kennes [16]. A pignistic transformation works by distributing evenly the mass of a belief function assigned to a focal set on each member of the set. This transformation has its root in Laplace's principle of insufficient reason. Under this hypothesis it has been shown [6] that given a pignistic probability distribution the least informative belief function that produces the pignistic probability is a consonant belief function whose focal elements are nested. The plausibility function of a consonant belief satisfies the axioms of possibility theory. Thus, under the credal-pignistic hypothesis and the principle of least commitment, subjective possibility as an internal epistemic state can be inferred from the revealed betting behavior.

An important distinction between DPS and our approaches is that the "credal-pignistic" hypothesis which plays a critical role in DPS is rejected in our approach. An argument against the hypothesis is that it cannot account for the well-documented effects of ambiguity on decision making because decisions are made based on pignistic probability. Another difference between the two approaches is that while both credal belief function and the associated pignistic probability are two measures of uncertainty on the same domain, our approach deals with different variables on different domains.

A decision theoretic foundation, in style of Savage, for qualitative possibility theory has been studied in [5], [19] and [4]. Dubois, Prade and Sabbadin consider two decision criteria to compare acts.

$$v_*(f) = \inf_{s \in \Omega_H} \max(n(\Pi(s)), \mu(f(s))) \tag{37}$$

$$v^*(f) = \sup_{s \in \Omega_H} \min(m(\Pi(s)), \mu(f(s))) \tag{38}$$

where Ω_H is the domain of the variable, Π is the possibility distribution, f is an act (mapping from Ω_H to set of prizes), μ is the normalized utility function of the prizes, n is a order reversing function from the uncertainty scale into the utility scale and m is order preserving function. The *pessimistic* criterion (v_*) bears a similar idea to the minimax rule while the *optimistic* criterion (v^*) is similar to the maximax rule.

They ask what are the conditions a preference relation on acts must satisfy so that it can be represented by the pessimistic criterion (the optimistic criterion). We rewrite the conditions (theorem 5 and 6 in [5]) in the notation used in this paper. Besides technical conditions that $\succeq_{DPS}$ is a weak order (Sav1 in [5]) and there exist distinct prizes x, y such that $x > y$ (Sav5), three informative conditions are as follows.

WS3 Weak coherence with constant acts. $x, y \in \mathcal{O}$, h is an act, $A \subseteq \Omega_H$ and $x \geq y$ then $[A \mapsto x, \bar{A} \mapsto h] \succeq_{DPS} [A \mapsto y, \bar{A} \mapsto h]$.

PES Pessimism. $[A \mapsto f, \bar{A} \mapsto g] \succ_{DPS} f$ then $f \succeq [A \mapsto g, \bar{A} \mapsto f]$

RDD Restricted disjunctive-dominance. $f \succ_{DPS} g$ and $f \succ_{DPS} x$ then $f \succ_{DPS} g \vee x$ where $g \vee x$ is a disjunctive act i.e., $(g \vee x)(s) = \max(g(s), x)$.

They shown that if preference $\succeq_{DPS}$ satisfies Sav1, Sav5, WS3, PES and RDD then there exists a qualitative scale Q and a possibility measure $\Pi : \Omega_H \to Q$ and a utility function $\mu : \mathcal{O} \to Q$ such that $f \succeq_{DPS} g \Leftrightarrow v_*(f) \geq v_*(g)$. A dual result is also obtained for the optimistic criterion by replacing the pessimism (PES) with its dual condition, the optimism (OPT), and replacing the restricted disjunctive-dominance (RDD) with its dual condition, the restricted conjunctive-dominance condition (RCD). A feature of this approach is commensurability between the uncertainty scale and the utility scale.

Unlike our approach, DPS characterization only deals with the ordinal (qualitative) aspect of possibility theory. As we have mentioned earlier, the max composition property of possibility is due to the reversibility between H and the ignorant variable while the partial reversibility with the roulette variable is responsible for the quantitative interpretation of possibility. A detailed discussion of DPS axiomatization can be found in [9]. In particular, the necessity to include the pessimism (optimism) axiom is questioned. It showed that the pessimistic and optimistic criteria are special cases of the *bipolar* criterion whose characterization needs neither pessimism nor optimism property. In [19], Weng proposed axiomatization for the bipolar utility criterion. Dubois, Fargier and Vantaggi [4] extended the axiomatization for conditional possibilistic preference. Finally, we note an argument in favor of the pessimistic (37) and optimistic (38) criteria because they could be more "discriminant" than the bipolar criterion. However, a closer look at the argument shows that the enhanced discriminating power is achieved at the cost of extra decision parameter (functions n and m in (37) and (38)). When the degrees of freedom in the criteria are controlled for, the apparent discriminating enhancement disappears.

To conclude this section, let us comment on variations of AA setup considered in literature, most notably, Schmeilder's model [14] that characterizes Choquet expected utility (CEU). In our work we follow the AA original axiomatization that includes a critical assumption of Reversal of order. We have shown that while it holds for variables of the same (uncertainty) type, it is impossible to maintain the property in the environment that mixes non-probabilistic uncertainty and risk. However, in [14] Schmeidler used a setting which traces back to Fishburn [7] in which the Reversal of order assumption is removed and replaced by a definition for mixture of horse acts. Suppose that f and g are horse acts then their mixture $\alpha f + (1-\alpha)g$ is defined pointwise i.e., for each outcome s of the horse race

$$(\alpha f + (1-\alpha)g)(s) \equiv \alpha f(s) + (1-\alpha)g(s). \tag{39}$$

Strictly speaking, the mixture $\alpha f + (1-\alpha)g$ is a lottery associated with a roulette spin (with probability α of for event Q) that delivers f if Q obtains, and g otherwise. In other words, after the spin, either f or g is excluded and becomes irrelevant. So before the horse race takes place the owner can sell her holding for a price. However that interpretation is not compatible with the definition (39) which insists that the mixture is equivalent to a compound lottery where the horse race takes place first and after that the roulette is spun. This is essentially the assumption of Reversal of order. To avoid the consequence of AA theorem that the uncertainty about the horse race must be subjective probability, Schmeidler gave up the assumption that preference over roulette lotteries is described by expected utility and replaced it with a weaker version of "Independence" axiom called "Comonotonic independence". It says that if f, g and h are pairwise comonotonic then

$$f \succeq g \text{ iff } \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h. \tag{40}$$

As a result, the horse uncertainty is characterized by a non-additive uncertainty (capacity) and the horse lotteries are evaluated by Choquet expected utility. A function v on the algebra of subsets of S is a capacity if $v(\emptyset) = 0$; $v(S) = 1$ and if $A \subseteq B$ then $v(A) \leq v(B)$. Thus, a possibility measure is a special capacity with additional structure $v(A \cup B) = \max(v(A), v(B))$. From a pragmatic point of view, the lack of structure in general capacity allows it to account for ambiguity effect but it makes inference difficult. For example, if $v(A), v(B)$ and $v(C)$ are known, nothing can be said about relationship between $v(A \cup B)$ and $v(A \cup C)$. While both Schmeidler's model and our model are deviations from the original AA model, we keep the expected utility assumption for roulette lotteries and give up the Reversal of order while Schmeidler did the opposite.

A popular model for uncertainty is the multiple priors model with associated Maxmin expected utility (MMEU) theory [10]. In this model, the axiom of comonotonic independence is further weakened to apply only for mixtures with a constant act (in (40) h must be a constant act). Another property imposed on preference is called "Uncertainty aversion": if $f \sim g$ then $\alpha f + (1-\alpha)g \succeq f$. Gilboa and Schmeidler have shown that if the preference satisfies those assumptions then there must exist a set of probability distributions D so that the preference over the horse lotteries is given by

$$f \succeq g \text{ iff } \min_{p \in D} E_p[u(f)] \geq \min_{p \in D} E_p[u(g)] \tag{41}$$

We argue that the MMEU model is subsumed in our model that consists of a variable of ignorance I preceding a variable of probability R. The domain of I is the set of indices $\{t|p_t \in D\}$, one for each probability measure in D. The conditional risk given $I = t$ is p_t. You can think of $\{t|p_t \in D\}$ as a set of roulette tables and each of the tables has a specific probability distribution. Our ignorance is about which table will be selected to play the roulette game. In our model, the preference between lotteries is described by

$$f \succeq g \text{ iff } \gamma_\tau\{E_p[u(f)]|p \in D\} \geq \gamma_\tau\{E_p[u(g)]|p \in D\} \tag{42}$$

where u is the vNM utility function representing the preference on roulette lotteries, normalized to the unit interval, $\gamma_\tau\{X\}$, a function on sets, returns a value closest to τ within the interval $[\min(X), \max(X)]$. In this sense, MMEU is a special and extreme case when $\tau = 0$. By varying the index of tolerance for ignorance τ, (42) can describe not only uncertainty aversion but also uncertainty seeking attitude. Moreover, in our model the uncertainty attitude no longer limits to a dichotomous classification of aversion vs seeking but it is possible to express the idea of relative uncertainty attitude, namely, the notion that an individual is more uncertainty averse than another individual.

8. Conclusion

This paper presents a new approach to characterize subjective possibility that extends Anscombe-Aumann's original approach to definition of subjective probability. It characterizes the uncertainty of a variable in relation to a variable of ignorance and a variable of probability. The idea of reversibility between variables plays a critical role in this characterization. Assuming Monotonicity axiom, Anscombe-Aumann showed that if a variable is completely reversible with respect to another probabilistic variable then it must be probabilistic too. Our main result says that the subjective uncertainty of a variable must obey the axioms of possibility theory if it is completely reversible wrt the ignorant variable and is partially reversible wrt the probabilistic variable. In particular, we derive a utility function for acts under subjective possibility uncertainty.

Conceptually, our work sheds new light on the material role of the order of variables in decision making that involves different types of uncertainty. If two uncertain variables are not reversible then their order of realization can change the certainty equivalent of a functional act contingent on their outcomes. In particular, this statement applies for the case of a probabilistic variable and a possibilistic variable. A surprising consequence of non-reversibility is that it excludes the traditional one-stage formulation of acts involving both risk and possibility uncertainty because such a formulation implicitly assumes reversibility.

Pragmatically, we prescribe a procedure by which subjective possibility can be extracted from observed choice behavior. Our axiomatization helps to answer a question of practical importance: whether or not possibility theory is appropriate for the subjective uncertainty under consideration. The answer depends on the testing of two conditions: the complete reversibility between that uncertainty and ignorance (RHI) and partial reversibility between it and probability (PRHR) are satisfied by individual behavior.

Finally, this work provides a framework for decision making in the environment of heterogeneous uncertainty that includes risk, possibility uncertainty and ignorance.

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